THEORETICAL AND PRACTICAL

MECHANICS,

DESIGNED

PRINCIPALLY FOR PRACTICAL MEN.

BY JAMES HANN, A.I.C.E.

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AUTHOR OF "A TREATISE ON THE STEAM ENGINE."

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1848.

TO THE

REV. T. G. HALL, A. M.

PROFESSOR OF MATHEMATICS, KING'S COLLEGE, LONDON.

REVEREND AND DEAR SIR,

I BEG to inscribe this Work to you, not only as a tribute of respect for your high attainments in these branches of science, but also as a grateful acknowledgment of the many kindnesses I have received from you during the series of years that I have been connected with this Institution.

Hoping that your latter years may be as tranquil and happy, as your former ones have been useful and successful,

I subscribe myself,
With every sentiment of respect and esteem,
Your faithful Friend,
And humble Servant,
JAMES HANN.

PREFACE.

In 1833 the "*Mechanics for Practical Men*" appeared, which I wrote in conjunction with my excellent friend, Mr. Isaac Dodds, the eminent engineer of Glasgow. This book, being of a more practical nature than almost any other work of the kind at that time, had a most rapid sale. The various avocations of my friend having precluded the possibility of writing for a new edition, he has, with that kindness for which he is as distinguished as for his great abilities, allowed me the full and free use of the above-named work: I have therefore borrowed largely from it. Many of the parts might be done more simply; and, in fact, I have not confined myself to the methods there given alone, but have extended them considerably: the greater part, however, of the mechanical powers is essentially the same.

For the Lever, when the forces act obliquely, I have given a geometrical construction, and also an algebraical solution. At page 82, an example is constructed by my talented friend and colleague, Mr. Haddon, with such accuracy that it agrees almost exactly with the algebraical solution.

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[illegible] used Petit's Tables, [illegible] horizontal thrust to the [illegible] moment of the hori- [illegible] of the semi-arch and [illegible] for stability, the [illegible] must be greater than

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[illegible] Mr. [illegible] Dr.

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Many of the problems are taken from the various periodicals, and from the Cambridge problems.

The works that I have consulted are those of Poncelet, Moseley, Tate, and those of my late lamented friend, Dr. Olinthus Gregory, whose *Mechanics* was undoubtedly the best work for practical purposes that had appeared before the works of Poncelet and Moseley.

The Tables on Iron, at the end, were made by my old pupils, Messrs. Setree and Shelley. The Parallel Motion Tables were made by my eldest son.

J. HANN.

KING'S COLLEGE SCHOOL,
October 23*d*, 1848.

CONTENTS.

ERRATA.

Page 20, line 8 from bottom, for AC sin. $\times$ $C\dot{A}\dot{D}$, read $AC \times$ sin. CAD.
— 21, — 4 from top, for $P + W$, read $P - W$.
— 26, — 14, *for* Art. 47, *read* Art. 26.
— 29, — 1, *for* Art. 47, *read* Art. 26.
— 63, — 1, *for* page 24, *read* page 25.
— — — 14, *for* page 46, *read* page 55.
— 73, — 5 from bottom, *read* $P = Q \cdot \frac{b + fr}{a - fr}$.

THEORETICAL & PRACTICAL MECHANICS

MECHANICAL DEFINITIONS.

1. MECHANICS is a science which teaches the properties of the forces, motions, velocities, and, in general, the actions of bodies upon one another.

2. Body is the mass, or quantity of matter, in any material substance; and it is always proportional to its weight or gravity, whatever its figure may be.

Body is either hard, soft, or elastic. A hard body is such that its parts do not yield to any stroke or percussion, but retain their figure unaltered. A soft body is such that its parts yield to any stroke or impression without restoring themselves again, the figure of the body remaining altered. And an elastic body is such that the parts will yield to any stroke, but which presently restore themselves again, and the body regains the same figure as before the stroke.

There are no bodies that are perfectly hard, soft, or elastic; but all partaking these properties, more or less, in some intermediate degree.

3. Bodies are either solid or fluid. A solid body is such that its parts are not easily moved among one another, and which retains any figure given to it. But a fluid body is such that the parts yield to the slightest impression, being easily moved among one another; and its surface, when left to itself, is always observed to settle in a smooth plane at the top.

[illegible]

[illegible] Absolute Gravity [illegible] space; and [illegible] descend with [illegible]

13. Specific Gravity [illegible] different bodies of equal magnitude [illegible] to the density of the body.

14. Centre of Gravity [illegible] upon which, the body [illegible] in any position.

15. Centre of M[illegible] which the body is moved [illegible] fixed axis it moves about.

16. Weight and Power, when opposed to each other, signify the body to be moved and the body that moves it. That body which communicates the motion is called the power; and that which receives it, the weight.

17. Statics has for its object the equilibrium of forces applied to solid bodies.

18. By Dynamics* we investigate the circumstances of the motion of solid bodies.

* The term Dynamics signifies literally the doctrine of power; power or force being known to us only as the cause of motion, and measured by the motion it produces.

4. Density of a body is the proportion of the quantity of matter contained in it, to the quantity of matter contained in another body of the same size or magnitude. Thus, the density is said to be double or triple, when the quantity of matter contained in the same space is double or triple.

5. Force is a power exerted on a body to move it. If it act but for a moment, it is called the force of percussion or impulse. If it act constantly, it is called an accelerative force. If constantly and equally, it is called an uniform accelerative force.

6. Velocity is an affection of motion, by which a body passes over a certain space in a certain time. Thus, if a body in motion pass uniformly over 10 feet in 2 seconds of time, it is said to move with the velocity of 5 feet per second; and so on.

7. Motion is a continual and successive change of place. If a body moves through equal spaces in equal times, it is called equable motion. If its velocity continually increases, it is called accelerated motion. If it decreases, it is retarded motion. If it increases or decreases uniformly, it is equably accelerated or retarded. Likewise if its motion be considered in regard to some other body at rest, it is called absolute motion; but if its motion be considered with respect to other bodies also in motion, then it is relative motion.

8. Direction of motion is the way the body tends, or the right line it moves in.

9. Momentum, or Quantity of Motion, is the power or force in moving bodies by which they continually tend from their present places, or with which they strike any obstacle that opposes their motion.

10. Forces are distinguished into Motive, or Accelerative or Retardive. A Motive, or moving force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in a body, when acting either by percussion, or for a certain time as a permanent force.

11. Acceler[illegible]
stood t[illegible]
by which [illegible]
equal or propor[illegible]
the mass [illegible]
pounds weig[illegible]

accelerative [illegible] =

on another [illegible]

force in this ca[illegible] =

and will produce [illegible]

12. Gravity is [illegible]
to descend towards [illegible]
Absolute Gravity [illegible]
space; and Relati[illegible]
descend with [illegible]

13. Specific Grav[illegible]
different bodies of equal [illegible]
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14. Centre of Gravity [illegible]
upon which, the body being [illegible]
in any position.

15. Centre of Motion [illegible]
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16. W[illegible] and Pow[illegible]
signify the body to be [illegible]
That body which communicates [illegible]
power [illegible] that w[illegible]

17. Statics [illegible]
applied to so[illegible]

18. By [illegible]
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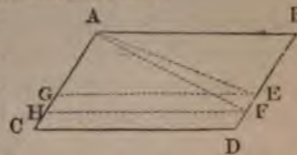
replaced by their resultant. By supposition this resultant will have its direction in AF, and will therefore produce the same effect as though applied at F. But thus applied at F, it may in its turn be replaced by two other pressures acting in CF and BF, and these will manifestly be equal to P and Q, of which P may be transferred without altering the conditions of the equilibrium to C and Q to E. We have now then, without altering the conditions, replaced the original pressures by P and Q' acting in CF and CD at C and Q in FE at E. Now by supposition the resultant of P and Q' acts in CE, let them be replaced by this resultant; and let this resultant again be replaced by its components in DE and FE; these last will manifestly equal P and Q'. So that on the whole we have now, without altering the conditions, replaced the forces P, Q, Q' acting at A, by equal forces P, Q, Q' in parallel directions acting at E; or, in other words, the forces P, Q, and Q' produce the same effect whether applied at A or E. The fourth pressure which will hold these in equilibrium must therefore be such as will hold them in equilibrium when they are applied at A or at E; it must therefore evidently act in the line AE; now the resultant of the forces P, Q, and Q' is opposite to this pressure. This resultant acts therefore in the line AE. Q and Q' acting in the same straight line are equivalent to $Q + Q'$. So that the resultant of P, Q and Q' is the same with that of P and $Q + Q'$. It follows then (as was to be proved) that if the resultant of P and Q have its direction in AE and that of P and Q in CE, the direction of the resultant of P and $Q + Q'$ will be in AE.

Now this is true for all values of P, Q and Q'. It is true then when $P = Q = Q'$. But in this case the resultant of P and Q is manifestly in AF, and that of P and Q in CE. Hence therefore it follows that in this case the resultant of P and $Q + Q'$, that is, of P and $2P$, is in the diagonal AE. Taking now $Q = 2P$ and $Q' = P$ since the resultant of P and $2P$ is in the diagonal, and also that of P and P, it

follows that the resultant of P and $3P$ is in the diagonal. And so of P and $4P$, P and $5P$, and generally of P and nP, and thus similarly too of mP and nP, where m and n are any whole numbers. So that the law is true of any two forces which are commensurable. This is moreover true when the forces are incommensurable.



Let AC and AB represent any two such incommensurable forces. Complete the parallelogram $ACDB$, then shall the resultant of AC and AB be in AD.

For if not let its direction be AE and draw EG parallel to CD. Divide AB into any number of equal parts, each being less than GC; and set off parts on $AC =$ to these from A towards C. One of the divisions of these will manifestly fall in GC. Let it be H. Complete the parallelogram $AHFB$. Then AB and AH representing two forces, and being commensurable, their resultant will be in AF, and will have its direction nearer to AC than the resultant AE of AB and AC has, which is absurd, since AH is less than AC. Therefore AE is not in the direction of the resultant of AB and AC, and in the same manner it may be shown that no other than AD is in that direction.

2. The resultant is represented in magnitude as well as in direction by the diagonal of the parallelogram. Let BA and CA represent two pressures in magnitude and direction, complete the parallelogram $BACD$, then from above DA will represent the direction of the resultant; it will also represent it in magnitude.

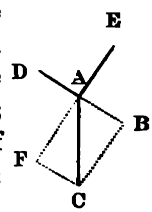
Produce DA to G, and apply in GA a pressure equal to the resultant and opposite to it, and let this pressure be

represented in magnitude by the line GA. Then will the forces BA, CA, and GA balance one another. Complete the parallelogram $GABF$, then is the resultant of GA and BA in the direction FA; but this resultant will manifestly balance CA; therefore FAC is a straight line, so that FA is parallel to BD, and $AFBD$ is a parallelogram.

Hence $AD = FB = GA =$ magnitude of resultant.

Therefore the resultant of CA and BA is represented in magnitude as well as in direction by the straight line DA.

3. If a body be kept in equilibrio by the joint action of three forces in the same plane, these forces will be respectively proportional to the three sides, AB, BC, CA, of a triangle, which are drawn parallel to the directions of the forces, DA, EA, CA.



Let AC represent the force C, and produce DA, EA, and complete the parallelogram $ABCF$. Now, by the last Prop. the force AC is equivalent to the two forces AB, AF; put, therefore, the forces AB, AF, instead of AC, and all the forces will still be in equilibrio: therefore, since AC represents the force C, then AB will represent its opposite force D, and BC or AF its opposite E. Consequently, the three forces, D, E, C, are proportional to AB, BC, AC, the three sides of the triangle ABC, formed by drawing lines parallel to the directions of the three forces.

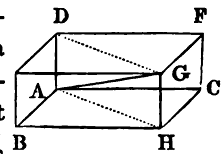
Cor. 1. The three forces D, E, C, will be respectively as the sines of the angles ACB, CAB, ABC; for these forces are as AB, BC, AC, and these sides are as the sines of their opposite angles C, A, B.

Cor. 2. Three forces acting upon a body, and keeping it in equilibrium, are proportional to the sides of a triangle formed by drawing lines either perpendicular to the directions in which the forces act, or making any given angles with those directions. For such a triangle is always similar

to that which is made by drawing lines parallel to the directions.

4. If three forces, the directions of which concur in one point, are represented by the three contiguous edges of a parallelopiped, their resultant will be represented, both in magnitude and direction, by the diagonal drawn from the point of concourse to the opposite angle of the parallelopiped.

Let the directions in which the forces act be AB, AC, and AD, and complete the parallelopiped BF. Then, since $ABHC$ is a parallelogram, the force AH is equivalent to the two forces AB, AC; but DG is both equal and parallel to AH,



and AD is both equal and parallel to GH; therefore $ADGH$ is a parallelogram, and a force which is represented by its diagonal AG, is equivalent to the two forces represented by AD, AH; that is, to the three forces AB, AC, AD.

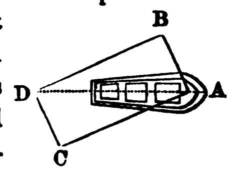
Cor. 1. If four forces, in different planes, act upon a body and keep it in equilibrio, these forces are to each other as the three edges and diagonal of a parallelopiped, constructed upon lines respectively parallel to the directions of the forces.

Cor. 2. It also follows, that a single force may be resolved into three others in different planes: also, each of these forces may again be resolved into others, either in the same or different planes; and so on, as far as we please.

5. The properties in the preceding propositions hold good for all similar forces whatever, whether they be instantaneous or continual, or whether they act by percussion, drawing, pushing, pressing, or weighing, and are of the utmost importance in mechanics and the application of the doctrine of forces to natural philosophy.

Ex. 1. Suppose a boat to be fastened to a fixed point by a rope, and acted on at the same time by the wind and the

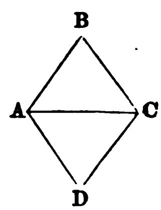
current; then the direction of the rope will represent the resultant of these forces. Thus, let AB and AC be at right angles, and the forces in these directions 30 pounds and 40 pounds respectively; required the magnitude and direction of the resultant.



Since the angle CAB is a right angle, and $AB=30$, $AC=40$, AD will be the resultant; and $AD^2=30^2+40^2=2500$, $\therefore AD=50$ pounds, the magnitude of the resultant. And to find the angle DAC, we have, $\sin. DAC=\frac{DC}{AD}=\frac{AB}{AD}=\frac{30}{50}=\frac{3}{5}=.6$. By the table of sines, .6 is the sine of $36^\circ\ 52'$.

Ex. 2. If two equal forces act at an angle of 120°; prove that the resultant or force compounded of them is equal to either of the equal forces.

Let AB and AD represent the two equal forces, and the angle $BAD=120^\circ$; then draw BC parallel to AD, and DC parallel to AB. Join AC; and since the angle $BAD=120^\circ$, the angle ABC, which is equal to the supplement of the angle BAD, must be equal to 60°; but $AB=BC$, therefore the angle BAC will be equal to the angle BCA. And since the three angles of the triangle ABC, taken together, are equal to two right angles, or 180°,

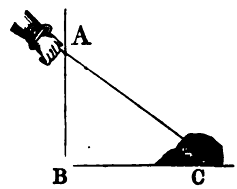


$$\text{we have } 60^\circ+\angle BAC+\angle BCA=180^\circ$$
$$\text{or } 60^\circ+2\angle BAC=180^\circ$$
$$\therefore 2\angle BAC=180^\circ-60^\circ=120^\circ$$
$$\therefore \angle BAC=120^\circ\div 2=60^\circ$$

Therefore since each angle of the triangle ABC is equal to 60°, the triangle is equilateral, or AB and BC are each equal to AC.

We may very frequently see examples of the resolution of forces, where the force exerted being resolved into two, one of them is totally lost or counteracted, and

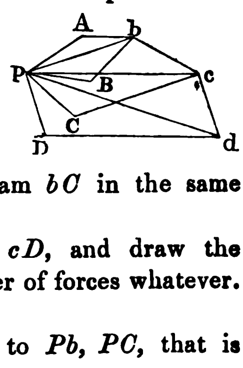
the remaining part only is effective. Thus, when we draw any body along the ground by a rope fastened to it, and supposing the rope to be inclined to the horizon at an angle of 45 degrees, the force which we exert is effective only in part. Thus, if we exert a force of 20 pounds, this is equivalent to two; one in the direction of AB, perpendicular to the horizon, and the other in the direction BC, parallel to the horizon; and $AB^2 + BC^2 = 20^2$; but since the angle $ACB = 45°$, $AB = BC$, $\therefore 2BC^2 = 20^2$, or $BC^2 = 400 \div 2$, therefore $BC = \sqrt{200} = 14.14$ pounds.



Hence the force with which we draw the body horizontally is 14.14 pounds.

6. To find the resultant of any number of forces concurring in one point P, and acting in the same plane.

Let the forces be represented by PA, PB, PC, PD, and from the point A draw Ab parallel and equal to PB, and complete the parallelogram AB, and draw the diagonal Pb. Then complete the parallelogram bC in the same manner, and draw the diagonal Pc.



Also complete the parallelogram cD, and draw the diagonal Pd; and so on for any number of forces whatever.

PA and PB are equivalent to Pb.

$\therefore$ PA, PB, PC, are equivalent to Pb, PC, that is to Pc.

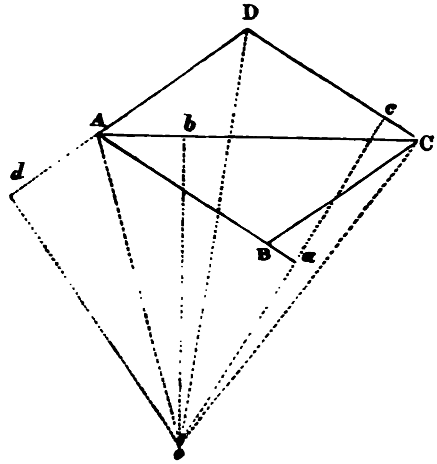
$\therefore$ PA, PB, PC, PD, are equivalent to Pc, PD, that is to Pd.

That is, Pd is the resultant of the forces PA, PB, PC, PD.

Cor. Hence it follows that if any number of forces be represented by the sides of a polygon taken in order, as PA, Ab, bc, cd, their resultant will be represented by the line Pd, that completes the polygon.

THE PRINCIPLE OF THE EQUALITY OF MOMENTS.

7. Let any point O be taken in the plane of the parallelogram $ABCD$: join OA, OD, and OC. Now, the quadri-



lateral $OADC$ is composed of the triangles AOD and ODC. Hence we have,

$$OAC = OAD + ODC - ADC \; . \; . \; . \; . \; (1)$$

From O let fall the perpendiculars Oa, Oc, Ob, and Od on the sides AB, AC, AD, or on those produced; then the area of the triangle $OAC = \dfrac{AC \times Ob}{2}$; the area

$$OAD = \frac{AD \times Od}{2};$$

$$ODC = \frac{DC \times Oc}{2};$$

$$ADC = \frac{DC \times ac}{2}.$$

Hence, by (1), we have $AC \times Ob = AD \times Od + DC \times Oc - DC \times ac$

$$= AD \times Od + DC\,(Oc - ac)$$

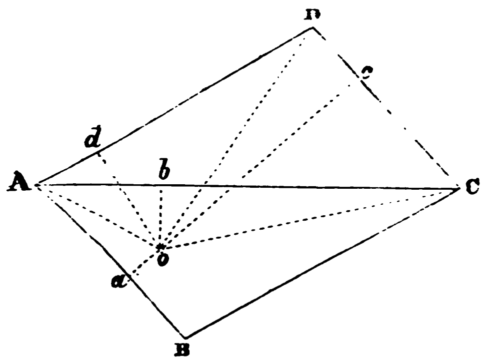
$$= AD \times Od + AB \, . \, Oa.$$

The products ($AC \times Ob$, $AD \times Od$, $AB \times Oa$) of the sides

AC, AD, AB, by the perpendiculars Ob, Od, Oa, let fall from the point O upon their respective directions, are called the moments. Hence we see that the moment of the resultant is equal to the sum of the moments of the components.

If the point O be taken within the parallelogram, we shall then have

$$OAC = OAD + ODC - ADC.$$



$$\text{Now } OAC = \frac{AC \times Ob}{2};$$

$$OAD = \frac{AD \times Od}{2};$$

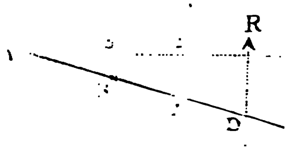
$$ODC = \frac{DC \times Oc}{2};$$

$$ADC = \frac{DC \times ac}{2}.$$

Consequently,

$$AC \times Ob = AD \times Od - AB \times Oa.$$

Hence, the moment of the resultant is equal to the sum of the moments of the components, which tend to turn the body in one direction, minus the sum of the moments of the components, which tend to turn the body in the opposite direction.



... to them at ...
... in each case
... multiplied by the sin ...
... Thus, $Ab = AB$ ...
... which is the same as AC ...
... and so on.

... find the resultant of any ...
... R be a force equal and ...
parallel forces, P, Q, S, T: ...
... to their directions till they ...
... equilibrium with the force ...

$$\ldots Q + S + T$$

$$\ldots \times P + AE \times S + A\ldots$$

$$\ldots \times S + AF \times T - A\ldots$$

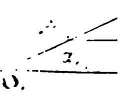
$$\ldots \times S + AF \times T - AC\ldots$$

$$R$$

$$\ldots AF \times T - AC \times G\ldots$$

$$\ldots Q + S + T$$

P



... α; P co...

and those parallel to AY are [illegible]
$x_i y_i$; $x_{ii} y_{ii}$ be the co-ordinates of [illegible]
$P_i \cos. a_i = X_i$, &c. and [illegible] &c.
$\overline{X}$ the resultant of the first set, and $\overline{Y}$ the resultant of the second; then

$$\overline{X} = X_i + X_{ii} + \&c.$$

$$\overline{Y} = Y_i + Y_{ii} + \&c.$$

If b be the distance of the resultant $\overline{X}$ from AX, and a that of $\overline{Y}$ from AY, then we have

$$\overline{X}b = X_i y_i + X_{ii} y_{ii} + \&c.$$

$$\overline{Y}a = Y_i x_i + Y_{ii} x_{ii} + \&c.$$

From these four equations we can determine the four quantities $\overline{X}$, $\overline{Y}$, a, b.

If R be the resultant of $\overline{X}$ and $\overline{Y}$, and a the angle it makes with the axis AX

$$R^2 = \overline{X}^2 + \overline{Y}^2;$$

$$\tan. a = \frac{\overline{Y}}{\overline{X}}.$$

Hence the magnitude and direction of the resultant are known.

10. If two forces P and Q meet in a point making an angle θ, find the resultant R and the angles α, β, which it makes with the components P, Q.

Let AB and AF (see fig. page 8), represent P and Q respectively, $BAF=\theta$, then

$$AC^2 = AB^2 + BC^2 - 2AB \times BC \cos. ABC,$$

or, $R^2 = P^2 + Q^2 - 2P \times Q \cos. ABC \quad \ldots \ldots (1)$

but $ABC = \pi - BAF = \pi - \theta$;

$\therefore \cos. ABC = \cos. (\pi - \theta) = -\cos. \theta.$

Substituting this in (1) we have

$$R^2 = P^2 + Q^2 + 2P \times Q \cos. \theta;$$

and since $BAC = \alpha$, and $FAC = \beta$,

$$\frac{\sin. BAC}{\sin. ABC} = \frac{Q}{R}, \text{ but } \sin. ABC = \sin. BAF.$$

Since the sine of angle is the same as the sine of the supplement

$$\frac{\sin.\ \alpha}{\sin.\ \theta} = \frac{Q}{R} \quad \therefore \ \sin.\ \alpha = \frac{Q}{R} \sin.\ \theta.$$

In the same manner,

$$\frac{\sin.\ \beta}{\sin.\ \theta} = \frac{P}{R} \quad \therefore \ \sin.\ \beta = \frac{P}{R} \sin.\ \theta.$$

If P and Q be equal, then

$$\alpha = \beta = \frac{\theta}{2} \text{ and } R = 2P \cos.\alpha;$$

for $R^2 = 2P^2 + 2P^2 \cos.\ \theta = 2P^2 + 2P^2 \cos.\ 2\alpha$

$$= 2P^2 + 2P^2 (2 \cos.^2 \alpha - 1) = 4P^2 \cos.^2 \alpha.$$

$$\therefore R = \sqrt{4P^2 \cos.^2 \alpha} = 2P \cos.\ \alpha.$$

If the forces meet at right angles, then

$$R = \sqrt{P^2 + Q^2};$$

$$\sin.\ \alpha = \cos.\ \beta = \frac{Q}{R};\ \sin.\ \beta = \cos.\ \alpha = \frac{P}{R}.$$

11. If three forces P, Q, R, meet in a point at right angles, we can determine the resultant and the angles α, β, γ, which it makes with the three forces.

$\sqrt{P^2 + Q^2 + R^2}$ = the resultant;

$$\cos.\ \alpha = \frac{P}{\sqrt{P^2 + Q^2 + R^2}};\ \cos.\ \beta = \frac{Q}{\sqrt{P^2 + Q^2 + R^2}};$$

$$\cos.\ \gamma = \frac{R}{\sqrt{P^2 + Q^2 + R^2}};$$

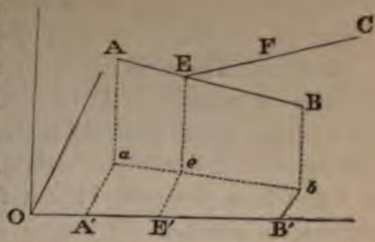
Square these three equations and add

$$\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = \frac{P^2 + Q^2 + R^2}{P^2 + Q^2 + R^2} = 1.$$

This is a well known property in the geometry of three dimensions.

[illegible] Any number of parallel forces, and the co-ordinates of their points of application being given, to determine the co-ordinates of the centre of the forces.

Let the points $A, B, C \ldots$ be acted upon by the parallel forces $P, Q, R \ldots$; and the system being referred to three rectangular axes, let $x, y, z, x', y', z', x'', y'', z'', \ldots$ be the co-ordinates of those points;



then shall the co-ordinates of the centre of parallel forces be

$$X = \frac{Px + Qx' + Rx'' \ldots}{P + Q + R}; \quad Y = \frac{Py + Qy' + Ry'' \ldots}{P + Q + R};$$

$$Z = \frac{Pz + Qz' + Rz'' \ldots}{P + Q + R}.$$

First, let there be only two points A and B acted upon by the forces P, Q, and determined by the abscissas $OA' = x$, $OB' = x'$; let E be the centre of the two forces P, Q, and let the abscissa $OE = \lambda$ correspond to it;

$$\therefore \; P : Q :: BE : AE :: B'E' : A'E' :: x' - \lambda : \lambda - x.$$

$$\therefore (P + Q)\,\lambda = Px + Qx'.$$

Suppose now the point E and the third point C of the system to be joined, which is acted upon by the force R, and has, for its abscissa x''; if F be the centre of the two forces $P + Q$ and R, and if λ' be its abscissa, it may be shewn, as before, that $(P + Q + R)\lambda' = (P + Q)\,\lambda + Rx''$; or, substituting the preceding value of λ, $(P + Q + R)$ $\lambda' = Px + Qx' + Rx''$. Proceeding in the same manner, until we reach the centre of all the forces, to which the abscissa X corresponds, we shall have

$$(P + Q + R \ldots)\,X = Px + Qx' + Rx'' \ldots\ldots$$

$$\therefore \; X = \frac{Px + Qx' + Rx'' \ldots.}{P + Q + R \ldots.}.$$

Next, referring the position of the points successively to the axes of y and of z, we shall have, in a similar manner,

$$(P + Q + R \ldots.)\,Y = Py + Qy' + Ry'' \ldots\ldots$$

$$\therefore \; Y = \frac{Py + Qy' + Ry'' \ldots.}{P + Q + R \ldots.}.$$

$$(P + Q + R \ldots.)\,Z = Pz + Pz' + Pz'' \ldots\ldots$$

$$\therefore Z = \frac{Pz + Qz' + Rz''....}{P + Q + R....}.$$

Example. There is an engine-room of a stean which is rectangular. At 10 feet from the side, 6 f the end, and 5 feet from the bottom, is situated the gravity of the boiler, whose weight, water included, at 4 feet from the side, 11 feet from end, and 7 f bottom, is situated the centre of gravity of th engine weighing $\frac{1}{2}$ ton; at 9 feet from side, 7 end, and 3 feet from the bottom, is the centre o furnace &c. weighing $1\frac{1}{2}$ tons; at 5 feet fr 11 feet from end, and 10 feet from bottom gravity of cylinder, piston, &c. weighing 1 t what distances from the side, end, and botto of gravity of the whole is situated.

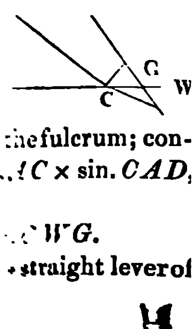
Let X, Y, Z = distances of centre of grav end, and bottom.

then $X = \dfrac{(10 \times 2) + (4 \times \frac{1}{2}) + (9 \times 1\frac{1}{2}) + (5\times 1)}{2 + \frac{1}{2} + 1\frac{1}{2} + 1}$ ticular CE is

$\dfrac{40\frac{1}{2}}{5} = 8.1$ feet from side.

$Y = \dfrac{(6 \times 2) + (11 \times \frac{1}{2}) + (7 \times 1\frac{1}{2}) + (11\times 1)}{2 + \frac{1}{2} + 1\frac{1}{2} + 1}$

$\dfrac{39}{5} = 7.8$ feet from end.



$Z = \dfrac{(5 \times 2) + (7 \times \frac{1}{2}) + (3 \times 1\frac{1}{2}) + (10\times 1)}{2 + \frac{1}{2} + 1\frac{1}{2} + 1}$ the fulcrum; con-

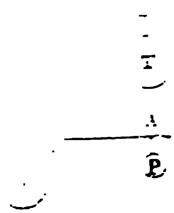
$\dfrac{28}{5} = 5.6$ feet from bottom. $C \times \sin. CAD$,

$W G$.

THE MECHANICAL PO straight lever of

13. The Mechanical Powers are con we are enabled to sustain a great weig great resistance, by a small force.

The mechanical powers are usually number; viz. the Lever, the Wheel an the Inclined Plane, the Wedge, and th

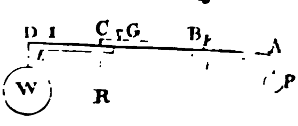


...crum, when the
...ether with the whole

...by Algebra. Thus, let
...lever $= a$, and $CW = x$;

..., as before.

...lever, or where several levers act per-
...another,
...fulcrums
... G, I; then
... $G \times DI : AF$



...er P acting at A : weight at B :: $BF : AF$,
...r power at B : the weight at C :: $CG : BG$.
...wer at C : the weight W :: $DI : IC$.
... ex equo, $P : W$ :: $BF \times CG \times DI$: AF

triangles CEB and CDF are similar; therefore $CD : CE :: CF : CB$.

Hence, by equality, $P : W :: CD : CE$.

That is, each force is reciprocally proportional to the distance of its direction from the fulcrum.

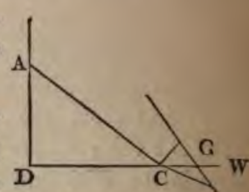
Cor. 1. When the two forces act perpendicularly on the lever, as two weights, then, in case of an equilibrium, D coincides with W, and E with P; therefore the above proportion becomes

$$P : W :: CW : CA.$$

Cor. 2. Since the product of the extremes is equal to the product of the means, $P \times CE = W \times CD$; or if the forces act perpendicularly on the lever, we have, by the last Cor. $P \times AC = W \times CW$.

Cor. 3. If any force P act at A in the direction AE, its effect on the lever to turn it about the centre of motion C, is as the length of the lever AC, and the sine of the angle of direction CAE. For the perpendicular CE is as $AC \times$ sine of the angle CAE.

Cor. 4. And in the bended lever ACW, we have $P \times AC \times \sin. CAD = W \times CW \times \sin. CWG$.

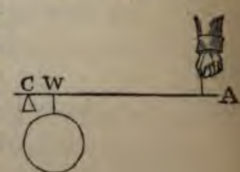


For CD and CG are perpendicular to the direction of the forces P and W respectively, and are therefore the distances of their respective directions from the fulcrum; consequently, $P \times CD = W \times CG$, but $CD = AC \times \sin. CAD$, and $CG = CW \times \sin. CWG$.

$\therefore P \times AC \sin. \times CAD = W \times CW \times \sin. CWG$.

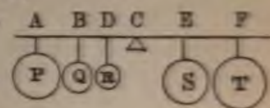
Cor. 5. It also follows, from Cor. 2, that in a straight lever of the first order, $P \times AC = W \times CW$, and the pressure on the fulcrum is $P + W$.

Cor. 6. In a straight lever of the second order $P \times AC = W \times CW$; but the pressure on the fulcrum is, in this case, $W - P$.



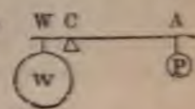
Cor. 7. In a straight lever of the third order, $P \times AC = W \times CW$, and the pressure on the fulcrum is $P + W$.

Cor. 8. If a straight lever be kept in equilibrio, by several weights P, Q, R, S, T, acting perpendicularly, then,



$P \times AC + Q \times BC + R \times DC = S \times EC + T \times FC$.

Cor. 9. Since $P : W :: CW : AC$, we have, by composition,



$$P + W : P :: AW : CW.$$

and $P + W : W :: AW : AC$.

$$\therefore CW = \frac{P \times AW}{P + W} \text{ and } AC = \frac{W \times AW}{P + W}.$$

This corollary is useful in finding the fulcrum, when the power and weight are both given, together with the whole length of the lever.

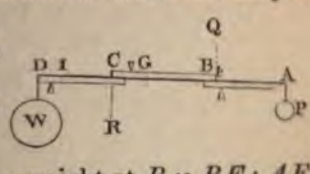
Or it may be done very simply by Algebra. Thus, let AW the whole length of the lever $= a$, and $CW = x$; then $AC = a - x$, and by Cor. 5,

$$Wx = P(a - x) = Pa - Px,$$

$$Px + Wx = Pa.$$

$$\therefore x = \frac{Pa}{P + W} = \frac{P \times AW}{P + W}, \text{ as before.}$$

20. In the compound lever, or where several levers act perpendicularly upon one another, as AB, BC, CD, the fulcrums of which are F, G, I; then $P : W :: BF \times CG \times DI : AF \times BG \times IC$.



For the power P acting at A : weight at B :: $BF : AF$, and the force or power at B : the weight at C :: $CG : BG$.

Also the power at C : the weight W :: $DI : IC$.

Therefore, ex equo, $P : W :: BF \times CG \times DI : AF \times BG \times IC$.

But it is more simple to divide the whole weight of the lever by the whole length, and the quotient will give the weight of one inch or one foot in length, according as you take the length in inches or in feet.

To apply the formulæ we have deduced, we may take Example 1.

Now, in this case, the power acts at one end of the lever, and the weight at the other, and the power is required; therefore formula (1) is exactly adapted to this case.

That is, $P=\dfrac{Wb+\frac{1}{2}b^2c-\frac{1}{2}a^2c}{a}$.

Here $W=100$ lbs. $a=60$ inches, $b=36$, and $c=\frac{4}{96}=\frac{1}{24}$ lb.; therefore,

$$P=\frac{100\times 36+\frac{1}{2}\times 36^2\times\frac{1}{24}-\frac{1}{2}\times 60^2\times\frac{1}{24}}{60}=$$

$\dfrac{3600+27-75}{60}=59.2$ lbs. the same as before in Ex. 1.

But if we take c for the weight of one foot in length, it becomes $P=\dfrac{100\times 3+\frac{1}{2}\times 3^2\times\frac{1}{2}-\frac{1}{2}\times 5^2\times\frac{1}{2}}{5}=$

$\dfrac{300+2.25-6.25}{5}=59.2$, as above.

We will now take Example 2d, where the power is applied at a given point D between A and C, and the weight at a given point E between W and C; and since the power P is required, we must take formula (8).

$$P=\frac{Wr+\frac{1}{2}b^2c-\frac{1}{2}a^2c}{d}.$$

Here $W=3$ cwt. $=336$ lbs. $a=8$ feet, $b=6$ feet, $c=\frac{42}{14}=3$ lbs. weight of one foot of the lever, $d=5$ feet, and $r=2$ feet; substitute these values in the above, and $P=\dfrac{336\times 2+\frac{1}{2}\times 6^2\times 3-\frac{1}{2}8^2\times 3}{5}=\dfrac{672+54-96}{5}=126$ lbs.

which is exactly the same as was found in Example 2d.

29. Let AWC be a lever of the second order, and C its fulcrum; the power multiplied by its distance from the fulcrum, is equal to the weight multiplied by its distance

from the fulcrum, together with the whole weight of the lever multiplied by half its length; the lever being considered uniform throughout its length.

Ex. 1. Given the whole weight of the lever 9 lbs. its length $AC = 6$ feet, a weight of 100 lbs. is put on at $1\frac{1}{2}$ feet from the fulcrum; it is required to determine the power acting at A which will keep the lever in equilibrio.

Now 100 lbs. $\times$ $1\frac{1}{2}$ feet $= 150 =$ the weight multiplied by its distance.

$9 \times 3 = 27 =$ the weight of the lever multiplied by half its length.

Hence $\frac{150 + 27}{6} = 29\frac{1}{2}$ lbs. is the weight or power acting at A which will keep the whole in equilibrio.

Or thus, by note to Art. 47:

$6 : 100 :: 4\frac{1}{2} : 75$, the weight upon the fulcrum from the action of the weight.

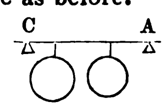
$6 : 100 :: 1\frac{1}{2} : 25$, the power at A which will just support the weight.

And the lever being uniform, its whole weight must be considered as acting at the middle of AC; therefore the fulcrum will bear one half of its weight, and the power must support the other.

Consequently $75 + 4\frac{1}{2} = 79\frac{1}{2}$ lbs. weight upon the fulcrum.

And $25 + 4\frac{1}{2} = 29\frac{1}{2}$ lbs. the power necessary to keep the whole in equilibrio, which is exactly the same as before.

Ex. 2. A beam, the weight of which is 12 lbs. and its length 18 feet, is supported at both ends; a weight of 36 lbs. is suspended at 3 feet from one end, and a weight of 24 lbs. at 8 feet from the other end; required the pressure on each prop or support.

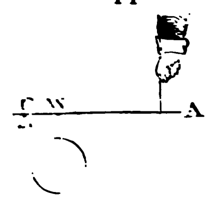


For the sake of simplicity, suppose the 36 lbs. weight to be suspended at 3 feet from the end C, and the 24 lbs. at 8 feet from the end A.

Then, $18 : 36 :: 15 : 30$ lbs. the pressure on the support C by the action of the 36 lbs. weight.

... the support C from
...
... the support C from both
...
... pressure on the support A from
...
... pressure on the support A from
...
... whole pressure on the support A
...
... of the lever added to each of the
... the whole pressure on each support.
... whole pressure on the support C.
... the whole pressure on the support A.

... whole length of the
... weight 15 lbs. a
... suspended at 2 feet
... what power,
... the other end A,
... equilibrio?



... the distance of the power from
... 30 :: 2 : 14⅔ lbs. the power
... alone; and since the centre
... from the power, and 5 feet
...
... necessary to sustain the lever.
... the power required to sustain
...
... and the weight sustained
...
... action of the weight.
... of the lever.
... whole pressure on the
...
... the lever of the
... as in Art. (27).
... we have

$$P = \frac{Wb}{a} + \frac{ca}{2}$$

$$W = \frac{Pa - \frac{1}{2}ca^2}{b} \quad (2)$$

$$a = \frac{1}{c}\left(P \pm \sqrt{P^2 - 2Wbc}\right) \quad (3)$$

$$b = \frac{Pa - \frac{1}{2}ca^2}{W} \quad (4)$$

$$c = \frac{2Pa - 2Wb}{a^2} \quad (5)$$

When $W = 0$, or the power just sustains the lever, we have $P = \frac{1}{2}ca$, and $a = \frac{2P}{c}$ (6)

If $P = 0$, the expression for a is impossible; therefore the equation for P admits of a minimum; and, by Dr. Gregory's Mechanics, Art. on the Lever, when P is a minimum, $a = \sqrt{\frac{2Wb}{c}}$.*

Take Example [illegible], and in that case $W = 100$, $a = 6$, $b = 1\frac{1}{2}$, and $c = [illegible]$; consequently we have, by formula [illegible],

$$P = \frac{100 \times 1\frac{1}{2}}{6} + \frac{6 \times [illegible]}{2} = \frac{150}{6} + \frac{[illegible]}{2} = 2[illegible] \text{ lbs.}$$

We will here give an example when a beam is supported by two posts or props, neither of which are at the ends of the beam.

Ex. 4. A beam, the weight of which is [illegible] lbs. and its length $AB = 20$ feet, is supported by two posts at C and D; the distance of the end A from $C = 1\frac{1}{2}$ feet, and the distance of the end B from $D = 2\frac{1}{2}$ feet; a weight of 1[illegible] lbs. is suspended at E, 2 feet from C; required the pressure on each post.

Now, $1\frac{1}{2} + 2\frac{1}{2} = 4 = AC + DB$, and $20 - 4 = 16 = CD$, the distance between the props or posts.

* $$\frac{dP}{da} = -\frac{Wb}{a^2} + \frac{c}{2} = 0.$$

$$\therefore \frac{c}{2} = \frac{Wb}{a^2}, \text{ or } a^2 = \frac{2Wb}{c}.$$

$$a = \sqrt{\frac{2Wb}{c}}.$$

$18 : 24 :: 8 : 10\frac{2}{3}$ lbs. the pressure on the support C the action of the 24 lbs. weight.

$30 + 10\frac{2}{3} = 40\frac{2}{3}$ lbs. pressure on the support C from weights.

And $18 : 36 :: 3 : 6$ lbs. pressure on the support A : the action of the 36 lbs. weight.

$18 : 24 :: 10 : 13\frac{1}{3}$ lbs. pressure on the support A i the 24 lbs. weight.

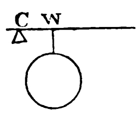
$6 + 13\frac{1}{3} = 19\frac{1}{3}$ lbs. the whole pressure on the suppo from both weights.

Now half the weight of the lever added to each of above sums will give the whole pressure on each suppc

Thus, $40\frac{2}{3} + 6 = 46\frac{2}{3}$ lbs. whole pressure on the suppo

And $19\frac{1}{3} + 6 = 25\frac{1}{3}$ lbs. the whole pressure on the suppc

Ex. 3. Given the whole length of the lever $AC = 10$ feet, its weight 15 lbs. a weight of 50 lbs. is suspended at 2 feet from the fulcrum or end C; what power, acting at 3 feet from the other end A, will keep the whole in equilibrio?



Now, $10 - 3 = 7$ feet, the distance of the power the fulcrum C; therefore $7 : 50 :: 2 : 14\frac{2}{7}$ lbs. the necessary to balance the weight alone; and since the of gravity of the lever is 2 feet from the power, and from the fulcrum, we have

$7 : 15 :: 5 : 10\frac{5}{7}$ lbs. the power necessary to sustain the

And $14\frac{2}{7} + 10\frac{5}{7} = 25$ lbs. the power required to both weight and lever.

In the same manner we may find the weight su by the fulcrum.

Thus, $7 : 50 :: 5 : 35\frac{5}{7}$ lbs. from the action of the w

And $7 : 15 :: 2 : 4\frac{2}{7}$ lbs. from the action of the lev

Therefore $35\frac{5}{7} + 4\frac{2}{7} = 40$ lbs. the whole pressure fulcrum or end C.

30. We may now deduce formulæ for the lever second order, retaining the same notation as in Art.

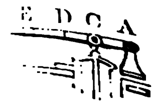
We have $Pa = Wb + \frac{1}{2}a^2c$, from which we have

.re

AD

and sin.

.5.



the valve $=AC=3$ inches, weight of the lever 4lbs. and the diameter of the valve 3 inches; what weight must be put on at B, the end of the lever, to give 40lbs. per square inch upon the valve?

Here $3^2 \times .7854 = 7.0686$, area, or number of square inches in the valve; but if we take 7 square inches, it will be near enough in practice.

Then $7 \times 40 = 280$lbs. the weight upon the whole valve and the weight of one inch in length is $\frac{4}{24} = \frac{1}{6}$lbs.

By Formula 2, in the lever of the third order,

$$W = \frac{Pa}{b} - \frac{bc}{2}.$$

Here $P = 280$lbs. $a = 3$, $b = 24$, and $c = \frac{1}{6}$lb.

$$\therefore \; W = \frac{280 \times 3}{24} - \frac{24 \times \frac{1}{6}}{2} = 35 - 2 = 33\text{lbs.}$$

That is, 33lbs. put on at the end of the lever, will give 280lbs. upon the whole valve, which is 40lbs. per square inch.

We must now mark the lever in the points where there will be 10, 20, and 30lbs. per square inch respectively.

To do this, we have the weight 33lbs. the distance $AC = 3$ inches, and when there are 10lbs. per square inch upon the valve, then $7 \times 10 = 70$lbs. upon the whole valve $= P$, to find what distance from the fulcrum the weight must be placed to give the above-mentioned pressure.

Here, in this case, Formula (10) is applicable.

$$r = \frac{Pa - \frac{1}{2}b^2c}{W} = \frac{70 \times 3 - \frac{1}{2} \times 24^2 \times \frac{1}{6}}{33} = \frac{210 - 48}{33} = 4.9$$

inches.

That is, if AD be taken $= 4.9$ inches, the weight of 33lbs. placed at D will give 70lbs. upon the whole valve, or 10lbs. per square inch.

In like manner, we must find how far we must move the weight along the lever from D towards B to give 20lbs. per square inch.

Here $7 \times 20 = 140$lbs. upon the whole valve; therefore

if we substitute 140lbs. for P in the above formula, we have

$$r = \frac{140 \times 3 - \frac{1}{2} \times 24^2 \times \frac{1}{6}}{33} = 11.27 \text{ inches} = AE.$$

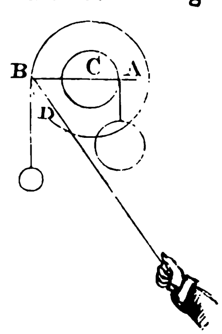
Also 30lbs. per square inch, multiplied by 7, gives 210lbs. the weight upon the whole valve; and this put for P, we have

$$r = \frac{210 \times 3 - \frac{1}{2} \times 24^2 \times \frac{1}{6}}{33} = 17.63 \text{ inches} = AF.$$

Therefore if you want 10, 20, 30, or 40lbs. per square inch upon the valve, the weight of 33lbs. must be at the distance of 4.9, 11.27, 17.63, 24 inches from the fulcrum A respectively.

THE WHEEL AND AXLE.

32. The wheel and axle consists of a wheel having a cylindric axis passing through its centre. The power is applied to the circumference of the wheel, and the weight to the circumference of the axle.



In the wheel and axle, an equilibrium takes place when the power multiplied by the radius of the wheel is equal to the weight multiplied by the radius of the axle; or $P : W :: CA : CB$.

For the wheel and axle being nothing else but a lever so contrived as to have a continued motion about its fulcrum C, the arms of which may be represented by AC and BC, therefore, by the property of the lever, $P : W :: CA : CB$.

33. If the power does not act at right angles to CB, but obliquely, draw CD perpendicular to the direction of the power, then, by the property of the lever, $P : W :: CA : CD$.

The capstan, the windlass, and various other contrivances of a similar nature, such as the gimlet and auger for boring holes, &c. may be referred to the same principle. Also,

the crank, as used in the steam engine, is a species of wheel and axle. If the force which acts upon a crank presses it directly up and down alternately, the effect, compared with what would take place if the force acted at right angles to the crank all round, is as twice the diameter of a circle to its circumference, or as 2 : 3.14159.

Therefore, to determine the mean length of crank, we must multiply the whole length of the crank by $\frac{2}{3.14159}$ or .6366; that is, when a given force is applied to a given crank as above, to raise a weight, the same effect will be produced if the force be applied at right angles to a crank the length of which is equal to the length of the given crank multiplied by .6366.

34. When two weights sustain each other by means of a wheel and axle, the thickness of the rope by which each weight is suspended must be taken into the account. We must add half the thickness to the distance at which P and W act respectively. Therefore if R = radius of the wheel, r = radius of the axle, and t = thickness of the rope, then we have $P : W :: r + \frac{1}{2}t : R + \frac{1}{2}t$.

$$\therefore P = \frac{W(r + \frac{1}{2}t)}{R + \frac{1}{2}t} \quad \ldots\ldots\ldots (1)$$

$$W = \frac{P(R + \frac{1}{2}t)}{r + \frac{1}{2}t} \quad \ldots\ldots (2)$$

$$R = \frac{W}{P}(r + \tfrac{1}{2}t) - \tfrac{1}{2}t \quad \ldots\ldots (3)$$

$$r = \frac{P}{W}(R + \tfrac{1}{2}t) - \tfrac{1}{2}t \quad \ldots (4)$$

35. If the wheel be acted on by a power instead of a weight, then the above proportion becomes $P : W :: r + \frac{1}{2}t : R$.

$$\therefore P = \frac{W(r + \frac{1}{2}t)}{R} \quad \ldots\ldots\ldots (5)$$

$$W = \frac{P \times R}{r + \frac{1}{2}t} \quad \ldots\ldots\ldots\ldots (6)$$

$$R = \frac{W(r + \frac{1}{2}t)}{P} \quad \ldots\ldots\ldots (7)$$

$$r = \frac{P \times R}{W} - \tfrac{1}{2}t \;\ldots\ldots\; (8)$$

36. In the case of two weights P and W, if the thickness of the rope by which P is suspended is not the same as the thickness of the rope by which W is suspended, then, if we put T for the latter thickness, and t for the former, we have

$$P : W :: r + \tfrac{1}{2}T : R + \tfrac{1}{2}t.$$

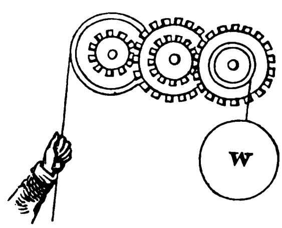
$$\therefore P = \frac{W(r + \frac{1}{2}T)}{R + \frac{1}{2}t} \;\ldots\ldots\; (9)$$

$$W = \frac{P(R + \frac{1}{2}t)}{r + \frac{1}{2}t} \;\ldots\ldots\; (10)$$

$$R = \frac{W}{P}(r + \tfrac{1}{2}T) - \tfrac{1}{2}t \;\ldots\; (11)$$

$$r = \frac{P}{W}(R + \tfrac{1}{2}t) - \tfrac{1}{2}T \;\ldots\; (12)$$

37. By a combination of wheels we may multiply the power to any extent whatever, by making the lesser wheels to turn the greater.



A combination of wheels and axles is the very same in principle as the combination of levers, given in Art. (20); therefore $P : W ::$ the product of the radii of all the axles : the product of the radii of all the wheels.

$$\text{Hence } P = \frac{W \times \text{the product of the radii of all the axles}}{\text{the product of the radii of all the wheels}} \;\ldots\; (13)$$

$$W = \frac{P \times \text{the product of the radii of all the wheels}}{\text{the product of the radii of all the axles}} \;\ldots\; (14)$$

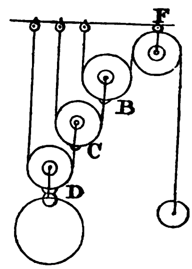
Or, since the number of teeth in wheels are as their radii,

$P : W ::$ the product of the number of teeth in all the pinions : to the product of the number of teeth in all the wheels.

$$\therefore P = \frac{W \times \text{the product of the number of teeth in all the pinions}}{\text{the product of the number of teeth in all the wheels}} \; (15)$$

$$W = \frac{P \times \text{the product of the number of teeth in all the wheels}}{\text{the product of the number of teeth in all the pinions}} \; (16)$$

the nth part of the weight, or $P : W :: 1 : n$. The pressure upon the hook A or $B = P + W$ or $(n+1)\,P$.



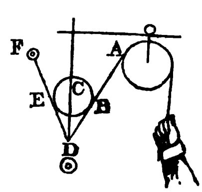
42. If, instead of the same cord going round all the pulleys, each pulley hangs by a separate cord, then, to obtain an equilibrium, $P : W :: 1 : 2^n$, n being the number of movable pulleys.

$$P : \text{weight at } B :: 1 : 2$$
$$\text{weight at } B : \text{weight at } C :: 1 : 2$$
$$\text{weight at } C : \text{weight at } D :: 1 : 2$$
$$\therefore\ P : W :: 1 : 2 \times 2 \times 2, \text{\&c.} :: 1 : 2^n.$$

Cor. Hence $2^n P = W$, or $P = W \div 2^n$.

43. When the strings or cords are not parallel to each other, but form the angle ADE, then $P : W :: 1$: twice the cosine of the angle of inclination of the direction of the power to the direction of the weight.

For produce AB, the direction of the power to D; and from C, the centre of the movable pulley, draw CB perpendicular to CD, the direction of the weight; then let DB represent the force in the direction DB, and resolve it into DC, BC, and DC will be that part of it which is effective in sustaining the weight; and since the cord EF sustains the same weight that the cord AB sustains, the whole weight sustained by the cord $EFBA$ will be represented by $2\,CD$, consequently $P : W :: DB : 2\,CD :: 1$: twice the cosine of the angle BDC.



Example. If a weight be sustained by a power which is attached to a rope going round a movable pulley, and making an angle of 30° with a vertical line passing through the centre of the pulley, what proportion does the power bear to the weight?

$$\text{By Art. 43, } P : W :: 1 : 2 \cos. 30^\circ;$$
$$\text{But } \cos. 30^\circ = \tfrac{1}{2}\sqrt{3};$$
$$\therefore\ P : W :: 1 : \sqrt{3}.$$

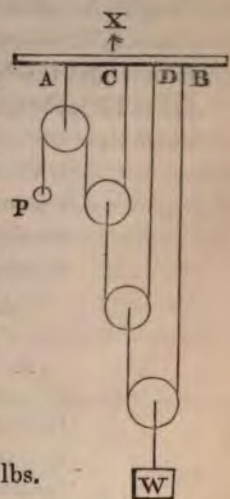
If the angle $BDC=45°$, then since $\cos. 45°=\frac{1}{2}\sqrt{2}$, we have $P:W::1:\sqrt{2}$.

If the angle $BDC=60°$, its cosine is $\frac{1}{2}$;

$$\therefore P:W::1:1;$$

Hence the power and weight are equal.

If the weight of the movable pulley be taken into consideration, then $P=\frac{W+w}{2\cos. BDC}$; w being the weight of the pulley.



There is a system of three movable pulleys and one fixed, as shown in the figure; the strings are attached to a rod AB, which is required to be supported by a single cord X; where must the cord be applied so that the bar may remain horizontal, all resistances being omitted, the weight $W=5$ cwt., diameter of each pulley $=8$ inches, and weight of each pulley $=3$ lbs.

$AC=12$ inches, $AD=16$ inches, $AB=20$ inches.

$$\text{tension at } B=\frac{560+3}{2}=281.5 \text{ lbs.}$$

$$\text{tension at } D=\frac{281.5+3}{2}=142.25 \text{ lbs.}$$

$$\text{tension at } C=\frac{142.25+3}{2}=72.625 \text{ lbs.}=P.$$

$$X=W+P+4\times 3=560+72.625+12=644.625;$$

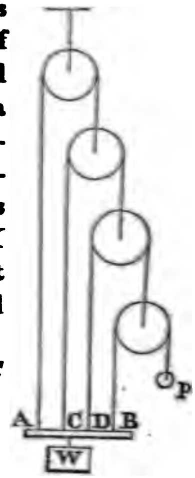
then, by the equality of moments, $AX\times X=AC\times C+AD\times D+AB\times B$;

$$\therefore 644.625\,AX=12\times 72.625+16\times 142.25+20\times 281.5$$

$$=871.5+2276+5630=8857.5.$$

$$\therefore AX=13.7 \text{ inches.}$$

There is a system of four pulleys, as shown in the figure, the extremities of the string passing over these being attached to a bar AB, from which is suspended a weight $W = 5$ cwt. to be raised; the pulleys are of the same weight and dimensions as in the preceding example. It is required to determine from what point X the weight W must be suspended so that the bar AB may remain in a horizontal position whilst the bar is being raised.



$AB = 12$ inches, $AD = 8$ inches, $AC = 4$ inches.

tension at $B = x$
tension at $D = 2x + 3$
tension at $C = 4x + 9$
tension at $A = 8x + 21$

$$\text{Now } 15x + 33 = 560;$$

$$\therefore\ x = \frac{527}{15} = 35.133.$$

tension at $B = 35.133$
tension at $D = 73.226$
tension at $C = 149.532$

$$AX \times W = AC \times C + AD \times D + AB \times B;$$

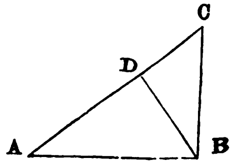
$$\therefore\ 560AX = 4 \times 149.533 + 8 \times 73.266 + 12 \times 35.133$$

$$= 598.133 + 586.133 + 421.599 = 1605.865;$$

$$\therefore\ AX = 2.8675 \text{ inches.}$$

THE INCLINED PLANE.

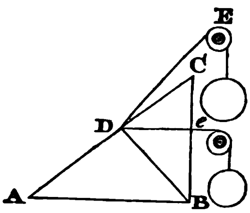
44. The Inclined Plane is a plane inclined to the horizon, or a plane which makes any angle whatever with an horizontal plane.



45. If a weight W be sustained upon an inclined plane by a power P, acting in a direction parallel to that

Or thus, by equ. 1, $P = \frac{600 \times 6}{10} = 360$ lbs. as before;

and by equ. 4, $P' = \frac{600 \times 8}{10} = 480$ lbs.



46. If the power, instead of acting in a direction AC, parallel to the plane, should act in a direction DE, making any angle EDC with it, then the power, weight, and pressure against the plane, are respectively as DE, EB, and DB;* for the weight W may be considered as kept in equilibrio by three forces acting in these directions.

Therefore the power $P : W :: DE : EB ::$ sin. DBE or sin. CAB† : sin. EDB.

P : pressure against the plane $P' :: DE : DB ::$ sin. CAB : sin. DEB.

Also, the weight W : pressure against the plane $P' :: EB : DB ::$ sin. EDB : sin. DEB.

$$\text{Hence the power } P = \frac{W.DE}{EB} = \frac{W.\sin. CAB}{\sin. EDB} \quad \ldots (1)$$

$$\text{Pressure } P' = \frac{P.DB}{DE} = \frac{P.\sin. DEB}{\sin. CAB} \quad \ldots (2)$$

$$\text{Also, } P' = \frac{W.DB}{EB} = \frac{W.\sin. DEB}{\sin. EDB} \quad \ldots (3)$$

Cor. 1. When the power acts in the direction De, parallel to the base of the plane, the three above proportions become

$P : W :: De : eB :: BC$‡ $: AB ::$ sin. CAB : cos. CAB.

$P : P' :: De : DB :: BC : AC ::$ sin. CAB : sin. 90.

$W : P' :: eB : DB :: AB : AC ::$ cos. CAB : sin. 90.

* DB is perpendicular to AC, [illegible] in last article.

† The triangles ACB and [illegible] angled triangles, and are evidently similar.

‡ The triangles A [illegible] similar.

$$P = \frac{W.BC}{AB} = \frac{W.\sin.CAB}{\cos.CAB} \ldots\ldots (4)$$

$$P' = \frac{P.AC}{BC} = \frac{P.\sin.90^\circ}{\sin.CAB} \ldots\ldots (5)$$

$$\text{Also } P' = \frac{W.AC}{AB} = \frac{W.\sin.90^\circ}{\cos.CAB} \ldots\ldots (6)$$

Cor. 2. The least power will be necessary to sustain a given weight when it acts in a direction parallel to the plane; for, by equ. 1, $P = \frac{W.\sin.CAB}{\sin.EDB}$, and since W and sin. CAB are both given, therefore P is proportional to $\frac{1}{\sin.EDB}$, and will evidently be the least possible when sin. EDB is the greatest; that is, when EDB is a right angle, or ED coincides with CD.

Cor. 3. The pressure against the plane will be greatest when the power acts in a direction parallel to the base of the plane; for, by equ. 2, $P' = \frac{P.\sin.DEB}{\sin.CAB}$, and supposing P and the sin. CAB given, then P' is proportional to sin. DEB, and is therefore greatest when sin. DEB is the greatest; that is, when DEB is a right angle, or when DE coincides with De.

Ex. 1. If a weight of 150 lbs. be sustained on an inclined plane by a power acting in a direction parallel to the base of the plane, the length of the plane being 10 feet, and the base 8 feet, required the pressure against the plane.

By Art. (46), Cor. 1, equ. 6, we have $P' = \frac{W \times AC}{AB}$.

Here $W = 150$, $AC = 10$ feet, $AB = 8$ feet; hence $P' = \frac{150 \times 10}{8} = 187\frac{1}{2}$ lbs.

Ex. 2. Compare the pressures against an inclined plane in the two following cases:

1st. When a body is sustained on an inclined plane by a ·er acting parallel to the plane.

THE WEDGE.

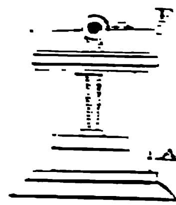
47. The Wedge is an instrument made of iron or some hard substance. Its form, in the most useful cases, is that of a prism contained between two isosceles triangles, as ACB.

48. In the wedge ACB, if the power acting perpendicularly to the back AB is to the force acting perpendicularly against either side AC or BC, as the breadth of the back AB is to the length of the side AC or BC, the wedge will be in equilibrio.

For, by Art. (3), Cor. 2, when three forces are in equilibrium, they are as the corresponding sides of a triangle drawn perpendicular to the directions in which these forces act. But AB is perpendicular to the direction of the force against the back, and AC, BC are perpendicular to the forces acting against them; therefore the three forces are as AB, AC, BC.

Cor. If we take into the account the resistance at both sides of the wedge, then, if there is an equilibrium, the power at D is to the whole resistance as the back AB is to the sum of the sides AC, BC, or as AB to $2AC$ or $2BC$.

In general, the wedge is used for splitting or cleaving wood, and separating the parts of hard bodies, by a blow from a hammer or mallet. The force impressed by percussion, or a blow on the back of the wedge, produces an effect incomparably greater than any dead weight or pressure, such as is employed in the other mechanical powers. And it may also be observed, that the wedge is seldom urged otherwise than by percussion; and very little can be gathered from the theory, but that the thinner the wedge is, the greater is its power.



[illegible] the base of the [illegible] cylinder, and [illegible] axis, an equili- [illegible] the weight or [illegible] threads of the [illegible] point to which

[illegible] inclined plane [illegible] has [illegible] of the [illegible] cylinder's [illegible] of the [illegible] threads: [illegible] the base, [illegible] the [illegible] circumference [illegible]

$=3$ feet, and the distance between two of the threads 2 inches; to find what weight a man would be able to sustain when he acts at P with a force of 150 lbs.

Now 6 feet $=72$ inches, and $72 \times 3.1416 = 226.1952$ inches $=$ the circumference described by the power; therefore, by Art. (50),

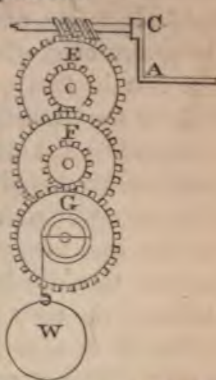
$$P : W :: 2 : 226.1952.$$

But $P = 150$ lbs. hence $W = \dfrac{226.1952 \times 150}{2} = 16964.64$ lbs.

When the screw is applied as in the figure below, it is called an endless screw; we have here also a combination of wheels, and, to obtain an equilibrium, we must have recourse to Art. (37), from which we have,

$P \times 2AC \times 3.1416 \times$ radii of all the wheels $= W \times$ distance of two threads $\times$ radii of all the pinions.

Ex. 2. If the endless screw be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turns a toothed wheel E, the pinion L of which turns another wheel F, and the pinion of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight a man will be able to sustain who acts at the handle CD with a force of 150 lbs., supposing the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches.

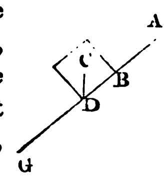


From what has been premised, we have $150 \times 40 \times 3.1416 \times 18^3 = W \times \frac{1}{2} \times 2^3$.

Hence $W = \dfrac{150 \times 40 \times 3.1416 \times 18^3}{\frac{1}{2} \times 2^3} = \dfrac{109930867.2}{4}$ $= 27482716.8$ lbs. or 12269 tons.

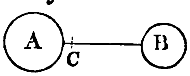
therefore all the force also with which it endeavours to descend.

Cor. 3. If a body be laid upon a plane GB, and one end A be gradually raised up, the body will slide down the plane if the perpendicular CD fall within the base; but if the perpendicular fall without the base, the body will roll down the plane.



It may be remarked here, that an equilibrium may take place when the centre of gravity is at the highest point to which it can ascend; but then this is only a tottering equilibrium, which the least motion will destroy; and the body or system, after the equilibrium is destroyed, will vibrate till the centre of gravity has obtained the lowest place of its descent.

54. The common centre of gravity C of any two bodies A, B, divides the line joining their centres into two parts, which are reciprocally as the bodies. $AC : BC :: B : A$.

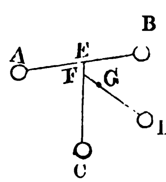


For if the centre of gravity C be supported, the two bodies A and B will be supported, and will rest in equilibrio. But by the property of the lever, when two bodies are in equilibrio about C as a fulcrum, we have $A \times AC = B \times BC$, or $AC : BC :: B : A$.

Cor. Hence, by composition, $A + B : B :: AB : AC$.

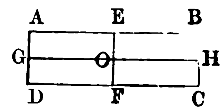
To find the centre of gravity of any number of bodies connected together by inflexible right lines without weight.

55. Let A, B, C, D, represent the bodies, the centre of gravity of which is required. Join any two of them, as A and B, by the right line AB; which divide in E, so that $A + B : B :: AB : AE$; and by Cor. to the last Art. E will be the centre of gravity of A and B. We must now suppose the sum of the bodies A and B to be collected in E, and join E, C, by the right



nd the centre of gravity of the area of a parallelogram.

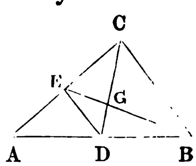
ıect AB in E, and AD in G; draw EF parallel ıd GH parallel to AB; their ɔn O is the centre of gravity For EF bisects every right can be drawn parallel to AB;



the centre of gravity must be somewhere in EF. bisects every right line that can be drawn parallel herefore the centre of gravity must be somewhere e GH. But it can only be in both these lines s at O, their point of intersection; therefore O ıtre of gravity of the area of the parallelogram

find the centre of gravity of a triangle.

t ABC be any triangle, and from any two of its and C draw the right lines CD o bisect the opposite sides in D hen will their intersection G be of gravity of the triangle. ce CD bisects AB, it will bisect



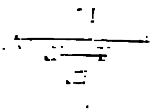
ht lines that can be drawn parallel to AB; that e parallel sections of the figure; therefore the gravity of the triangle lies in CD. For the on, it also lies in BE; consequently it is in G, mon point of intersection.

ın DE, which will be parallel to BC, and equal lf of it, and the triangles BGC and EGD are therefore, since $DE = \frac{1}{2} BC$, $DG = \frac{1}{2} CG$, or D. In the same manner, $BG = \frac{2}{3} BE$.

find the centre of gravity of a trapezium.

ide the trapezium $ABCD$ into two triangles by nal BD, and find the centres of gravity E and

[illegible] the [illegible] centre [illegible]



[illegible] that the centre [illegible]

[illegible] gravity [illegible]

[illegible] the centre of gravity [illegible] the triangles [illegible] then the centre of [illegible] in the line [illegible]

[illegible] that the centre of gravity of [illegible] since it lies both in [illegible] in their common [illegible]

[illegible] we may find the centre of gravity [illegible] for it may be divided [illegible] of gravity of which may be se- [illegible] that the centre of [illegible] the triangles, by considering the [illegible] centre of

[illegible] of a cylinder, or any other body [illegible] which are equal, is in the middle of [illegible]

[illegible] $\frac{1}{3}$ arc : sine of $\frac{1}{2}$ arc :: ra- [illegible] of gravity from the centre.

[illegible] arc : chord :: $\frac{2}{3}$ radius : [illegible] from the centre.

[illegible] the distance of the centre of [illegible] of the axis.

[illegible] of the surface of a cylinder, of [illegible] respectively at the

same distances from the origin as are the centres of gravity of the parallelogram, triangle, and trapezoid, which are vertical sections of the respective solids.

6. In a cone, as well as any other pyramid, the distance of the centre of gravity from the vertex is $\frac{3}{4}$ of the axis.

7. In a conic frustrum, the distance on the axis from the centre of the less end is $\frac{1}{4}h.\frac{3R^2+2Rr+r^2}{R^2+Rr+r^2}$, where h denotes the height, and R and r the radii of the greater and lesser ends.

8. The same theorem will serve for the frustrum of any regular pyramid, taking R and r for the sides of the two ends.

9. In the paraboloid, the distance from the vertex is $\frac{2}{3}$ of the axis.

10. In the frustrum of the paraboloid, the distance on the axis from the centre of the less end is $\frac{1}{3}h.\frac{2R^2+r^2}{R^2+r^2}$ where h denotes the height, R and r the radii of the greater and lesser ends.*

Ex. 1. Given the weights of two bodies 50 and 20lbs., and distance asunder 35 feet; how far from the larger body is their common centre of gravity?

By Art. (54), Cor. $50+20:20::35:10$ feet.

Therefore the centre of gravity is 10 feet from the larger body, and 25 feet from the smaller body.

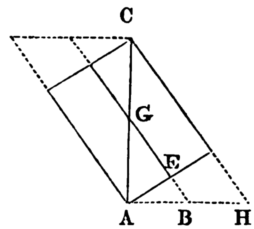
Ex. 2. If three equal bodies be placed at the angles of any triangle, prove that the common centre of gravity of these bodies is in the same point with the centre of gravity of the triangle.

The bodies A, B, and C, being all equal, the centre of gravity of any two of them, suppose A and B, will be in D, the middle of the side AB. The bodies A and B must now be supposed to be both collected in D, and join D, C, by the right line DC; and since $A+B=2C$, then, by

* The above are collected from the Mechanics of Emerson and Dr. Gregory.

the quotient will give the distance of the centre of gravity from the end where the weight is suspended.

Ex. 5. If the height of a cylinder be double the diameter of its base, what is the angle of inclination of its axis with the horizon when it is just ready to fall over?



By Art. (53), Cor. 1, the inclination of the axis of the cylinder must be such that a perpendicular, drawn from the centre of gravity G of the cylinder, will fall just on the extremity A of the base. Produce the axis of the cylinder to B; then the triangles GAB, AEB, and AEG, are all right-angled and similar, by Euclid, Book 6, Prop. 8; and since the altitude of the cylinder is double the diameter of its base, $GE=2AE$, consequently the angle $GAE=$ twice the angle AGE, whence the angle $GAE=60^\circ$, and the angle $AGE=30^\circ$; but the angle GAE is equal to the angle ABE, which is the angle of inclination of the axis of the cylinder with the horizon; therefore the axis makes an angle ABE of 60° with the horizon.

Also, since the angle AGE is equal to the angle BAE, the base of the cylinder will make an angle of 30° with the horizon.

If the cylinder be oblique, then CA will be equal to the altitude of the cylinder; but CA is equal to twice HA, therefore GA is equal to twice BA; hence the angle GBA is equal to twice the angle BGA. Therefore the angle GBA, which is the angle of inclination of the axis of the cylinder with the horizon, is 60°, the same as above.

Examples for Practice.

Ex. 1. A weight of $1\frac{1}{2}$ lbs. laid on the shoulder of a man, is no greater burden to him than its absolute weight, or 24 ounces; what difference will he feel between the said weight applied near his elbow, at 12 inches from the

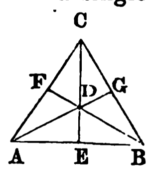
Calculation :—

$$\angle CAN = 180^\circ - \angle MAC$$
$$= 180^\circ - (25^\circ + 98^\circ)$$
$$= 57^\circ$$
$$AN = AC \times \cos. 57^\circ$$
$$= 10 \times .544639 = 5.44639$$
$$AM = AB \times \cos. 25^\circ$$
$$= 4 \times .9063078$$
$$= 3.6252312$$

The moment of Q about point A = the moment of P about A.

$$\therefore AN \times Q = AM \times P$$
$$\therefore 5.45 Q = 3.62 \times 28$$
$$\therefore Q = \frac{3.62 \times 28}{5.45}$$
$$= 18.6 \text{ lbs.}$$

Ex. 3. If a weight W be sustained on a horizontal plane by three props which are not in the same straight line, the pressure on each will be the same as if a single weight were laid on it, so that the sum of all the three weights were equal to W, and their common centre of gravity the same with the centre of gravity of that body.



If the props be A, B, C, and if W be placed with its centre of gravity at D, and if ADG, BDF, CDE, be drawn, then the pressure

$$\text{on } A = \frac{DG}{AG} \times W,$$

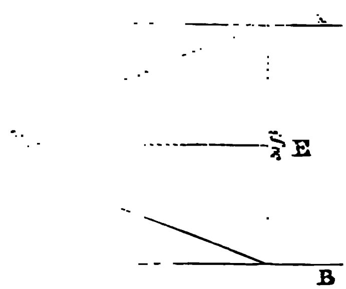
$$\text{on } B = \frac{DF}{BF} \times W,$$

$$\text{on } C = \frac{DE}{CE} \times W,$$

... with the centre of gravity of the ... the pressure on each of the props is the ... this case, DG is one-third of AG, DF is ... F, and DE is one-third of CE; therefore ... sustains one-third of the weight.

... be supported on more than two props, the ... appears to admit of innumerable solutions; but ... centre of gravity of W, as it rests on the plane, be ... the centre of gravity of the figure made by ... props by straight lines, the pressures ... equal to one another.

... dragged along a river by ropes ... opposite sides. The width of the ... each of the ropes is 72



... pulling with ... 7 cwt. on each ... the boat dragged?

... each equal 7, and complete ... the resultant measured ... 13.13 cwt. nearly.

... 4559 = 67.52

... 129 nearly.

... length, ... one of the ... the other

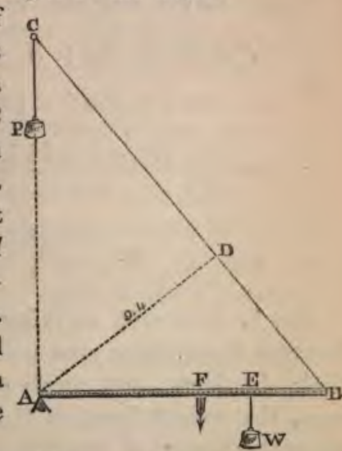
end to keep it parallel to the horizon; what is the weight of the beam?

Rule.—Multiply the weight which is suspended at the end by its distance from the fulcrum, and take twice this product for a dividend. Then subtract the square of the distance between the fulcrum and the end where the weight is suspended, from the square of the distance between the fulcrum and the other end, for a divisor. And if the above dividend be divided by this divisor, the quotient will give the weight of one foot or one inch in length, according as you take the length in feet or inches; and this quotient, multiplied by the whole length of the beam, will give the whole weight of the beam.

Here 2 feet is the distance between the fulcrum and the end where the weight is suspended, and 8 feet is the distance between the fulcrum and the other end.

By the rule, we have $\frac{2 \times 2 \times 224}{8^2 - 2^2} = \frac{896}{60} = 14\frac{14}{15}$ lbs. the weight of one foot in length of the beam; therefore $14\frac{14}{15} \times 10 = 149\frac{1}{3}$ lbs. the whole weight of the beam.

Ex. 6. A weight W of 2 cwt. is suspended 3 feet from the extremity B of a cylinder 12 feet long and 2 inches in diameter, of such quality as to weigh 485 lbs. per cubic foot. At the point C, 15 feet vertically above A is a pulley, and at A is a fulcrum on which AB turns. Supposing AB in a horizontal position, required to find a weight P that will balance the bar?



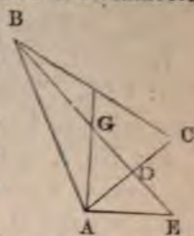
Construction:—Draw $AC = 15$ feet, join BC, and upon it let fall the perpendicular AD, which, measured from the same scale, will be 9.4 nearly, and since $AB = 12$ feet,

[redacted]

And by the note at page 24, we have 48 : 200 :: 30 : 125 lbs. the weight borne by the man who is nearest to the burden.

And 48 : 200 :: 18 : 75 lbs. the weight borne by the other man.

Ex. 8. If the altitude of a cone be double the diameter of its base, what is the inclination of its axis with the horizon when it is just ready to fall over?



By Art. (53), Cor. 1, when the cone is just ready to fall over, a perpendicular from the centre of gravity will fall on the extremity A of the base.

From the centre of gravity of a cone (page 46) we have $BD=4GD$, and by the question $BD=2AC=4AD$; hence $GD=AD$, and therefore the angles GAD and AGD are each 45°, or half a right angle. But the triangles AEG, GAD, and DAE, are all similar (Euclid, Book 6, Prop. 8); consequently, the angle AEG, which is the angle of inclination of the axis with the horizon, is 45°; and its complement, the angle DAE, which is the angle the base of the cone makes with the horizon, is also 45°.

Ex. 9. Two inclined planes AB and BC have the same height, and upon these planes two weights keep each other in equilibrio in the same manner as in Example 3, page 46; given the length of the planes 30 and 40 inches respectively, and the horizontal distance AC between the feet of the planes 50 inches; required their common height, and the ratio of the weights.

Rule.—As the base or longest side is to the sum of the other two sides, so is their difference to the difference of the segments of the base. And half the difference of the segments, added to half their sum, gives the greater segment; and half the difference of the segments, subtracted from half their sum, will give the lesser segment.

Thus, $50 : 40+30=70 :: 40-30=10 : 14$, the dif-

… ase: and $\frac{14}{2} = 7$, half the

… greater segment: hence 18

… common height of the planes.

… $V : 3C : 1B$.

… therefore $P : W :: 40$:

… must be in proportion to

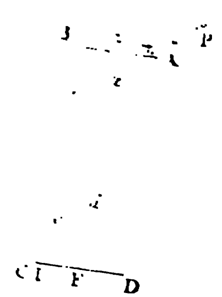
… uniform thickness, and in … upon two pins, one of … the other at the middle … on each pin when the … the middle of the base is … weight of the beam 56lbs. … centre of gravity is not in … page 54, the … from the vertex of a

… the centre of gra…

… the beam to … note, page 24, … the pin which

… the pin which

… ABCD … of a force … Pr… the force … call the perpen… … in order that … turn on the … BC



less than the weight of the pier multiplied by CF; $CF = \frac{1}{2}CD$, GH being the vertical through the centre of gravity of the pier.

This can be simply done by construction. Draw the pier to a scale, and measure off EC and CF; then $P \times EC < W \times CF$, but if $P \times EC = W \times CF$, then the pier will be on the point of turning on C. Or it may be done thus: take ad equal to the weight W of the pier, and ab equal to the pressure P; construct the parallelogram $abcd$; ac will represent the force tending to overturn the pier; and if ac produced meet the base between C and F the pier will stand, but if the point of intersection fall without the base, the pier will fall, and if it should intersect the base in the point C, the pier will be on the point of turning on C.

Ex. 12. Let the pressure P falling upon AB at the distance $An = \frac{3}{16}$ foot from A, make with AB the angle $PnA = 45°$; let $P = 2000$ lbs., the height $AD = 10$ feet, the breadth $CD = 4$ feet, the weight of a cubic foot of material $= 120$ lbs.

Then, by constructing the figure, EC is found to be 4.35.

$4.35 \times 2000 = 8700 =$ moment of pressure,

$1 \times 10 \times 4 \times 120 = 4800 =$ weight of wall; the length of the wall is here taken to be one foot.

$4800 \times 2 = 9600 =$ moment of wall; the moment of the wall being greater than the moment of the pressure, the wall will not be overturned.

Or, by the second method, taking ad, ab in thousandths, we have $ad = 4.8$, $ab = 2$, and by completing the parallelogram $abcd$, and drawing the diagonal ac, and producing it downwards, it will meet the base CD in I at the distance of $\frac{1}{3}$ foot from C on the inside, and consequently the pier will stand.

ference of the segments of the base; and $\frac{14}{2}=7$, half the difference of the segments.

Therefore $25+7=32$, the greater segment; hence is the lesser segment.

Then $\sqrt{40^2-32^2}=24$, the common height of the plan

And, by Example 3, page 46, $P : W :: BC : AB$.

Here $BC=40$, and $AB=30$; therefore $P : W :: 40 : 30 :: 4 : 3$. That is, the weights must be in proportion to each other as 4 to 3.

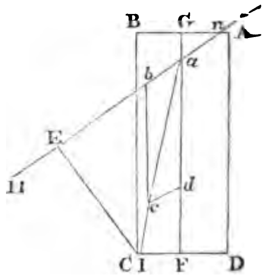
Ex. 10. A cast-iron beam, of uniform thickness, and the form of a parabola, is supported upon two pins, one of which is fixed at the vertex, and the other at the middle of the base; required the pressure on each pin when the distance between the vertex and the middle of the base is 5 feet, the base 2 feet, and the weight of the beam 56lbs.

Here, in this Example, the centre of gravity is not in the middle of the beam; and by Form. 4, page 54, the distance of the centre of gravity from the vertex of a parabola is $\frac{3}{5}$ of the axis.

Hence $\frac{3}{5}\times 5=3$ feet, the distance of the centre of gravity from the vertex.

Now we must consider the whole weight of the beam to be collected in its centre of gravity; and by note, page 5 we have $5 : 56 :: 3 : 33\frac{3}{5}$lbs. the pressure on the pin which is fixed in the middle of the base.

And $5 : 56 :: 2 : 22\frac{2}{5}$lbs. the pressure on the pin which is fixed in the vertex.



Ex. 11. To determine whether the pier $ABCD$ will overturn by the action of a force P in a given direction. Produce the direction of the force to H, and from C let fall the perpendicular EC; then, in order that the pier may not turn on the point C, we must have $P\times EC$

... the pressure P ... I at the distance ... make with ... $=$ ... let ... $=$ ... $=$ ... feet the ... of mate- ...

... the le- ...

... $=$ ... $=$ moment of pressure,

... $= 40$ ... $=$ weight of wall,

... $=$ ... $=$ moment of wall.

... taking ... in thousandths, $=$ ... completing the parallel- ... the diagonal ac and producing ... the base CD produced in I at ... the outside, and conse- ...

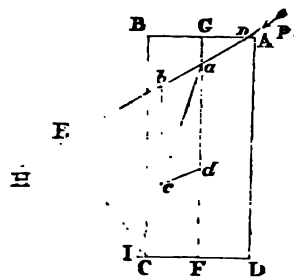
... falling upon AB at the ... make with AB the angle ... height $AD = 8$ feet, ... weight of a cubic foot of ...

... is found to be

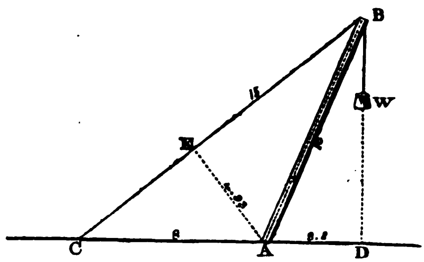
... pressure.

... wall.

... thousandths. ... the parallel- ... producing ... I at ... conse-



Ex. **15.** A weight W of 4 cwt. is suspended from the upper extremity of an iron bar AB, 10 feet long, resting upon the ground and supported by a string BC 15 feet



long, at such inclination that $AC=8$ feet. What is the amount of the tension produced by the weight W upon the string BC?

Construction:—From A draw $AC=8$ feet, from C with the distance $CB=15$ feet, describe an arc; and from A with the distance $AB=10$ feet, describe an arc intersecting the former in B, then AB will be the position of the bar; let fall the perpendicular AE on BC, and BD on AD, which being measured from the same scale, will give 4.93 and 3.8 respectively; then $4.93 \times \text{tension} = 3.8 \times W$.

Calculation:—

$$BC^2 = AB^2 + AC^2 + 2AC \times AD$$

$$\therefore AD = \frac{BC^2 - AB^2 - AC^2}{2AC}$$

$$= \frac{15^2 - 10^2 - 8^2}{2 \times 8} = \frac{61}{16}$$

$$= 3.8125$$

also in similar triangles AEC, BDC,

$$AE : AC :: BD : BC$$

and $BD = \sqrt{AB^2 - AD^2} = \sqrt{85.56} = 9.25$

will intersect each other in the point F; join $E F$ and $F A$; produce $F A$ to L: from G the centre of gravity of the cylinder, let fall the perpendicular $G K$, and from B the perpendicular $B L$; also, from A draw $A H$ perpendicular to $E F$; then $A H \times$ tension of the rope $= A K \times$ weight of cylinder $+ W \times A L$.

$2^2 \times .7854 \times 6 \times 140 = 3298.824$ lbs. = weight of cylinder.

$A H$, $A K$, and $A L$, when measured, will be found to be 4.44, .62, and 2.98 respectively.

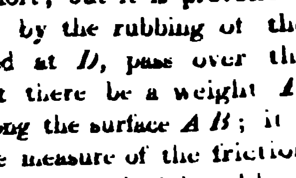
$$\therefore 4.44 \times \text{tension} = 3298.824 \times .62 + 448 \times 2.98$$

$$= 5142.17 - 1335.14 = [illegible] \text{ lbs.}$$

$$\therefore \text{tension} = \frac{[illegible]}{4.44} = [illegible] \text{ nearly.}$$

ON FRICTION.

60. Friction is proportional to the pressure. That is, every thing remaining the same, the friction increases as the pressure increases.



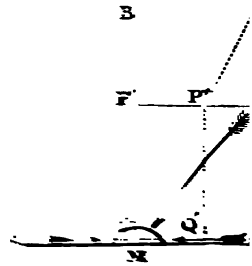
Thus, let the body M be placed upon the horizontal plane $A B$; since the weight of the body, M, is destroyed, it is evident, that abstracting from the resistance of the air, &c. the body ought to be put in motion by the slightest effort; but it is prevented by the friction which is caused by the rubbing of the surfaces. Let a string fastened at D, pass over the pulley C; at the other end let there be a weight P, sufficient to draw the body M along the surface $A B$; it is obvious, that this weight P is the measure of the friction, that is, if we double the weight of M, we shall be obliged to double the weight P; or if M be tripled, P must be tripled also. The power P then has, to the weight M a constant proportion, if we suppose that the surfaces do not change; and this is the sense in which we are to understand

THE LIMITING ANGLE OF RESISTANCE.

... has given in his Mechanics ...

... the following able exposition:—

... in the body M. Dr...



... which the surfaces are press...

... upon one another ...

... friction of this force P...

... friction of PM; comple...

... draw its diagonal P...

... friction of the body up...

... PM, whi...

... since, moreover, Q...

fraction $\frac{P'P''}{P'M}$, or $\frac{MQ'}{P'M}$, which is equal to that co-efficient. It is, therefore, the same for surfaces of the same nature, whatever be the actual amount of the impressed force P; but different for different surfaces.

From the theory of the inclined plane we can prove easily that $f = \tan.\theta$; θ being the inclination of the plane to the horizon.

For if a weight W be placed on an inclined plane; then $W \sin.\theta =$ component in the direction of the plane, and $W \cos.\theta$ component perpendicular to the plane; then $fW \cos.\theta =$ friction; now to find the inclination when the weight is just upon the point of moving, we have

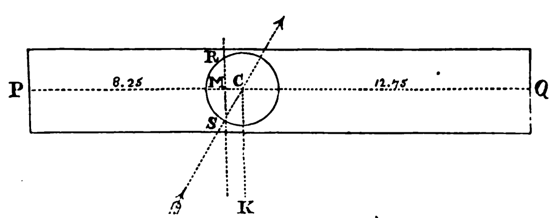
$$W \sin.\theta = f W \cos.\theta;$$

$$\therefore f = \frac{\sin.\theta}{\cos.\theta} = \tan.\theta.$$

There is a lever acted upon by two forces P and Q; in the middle of the lever is fixed at right angles to it an axis 3 inches in diameter. CP being 9 inches in length, CQ 1 foot, and the limiting angle of resistance 30°, required the relation between P and Q.

In order that motion may ensue, the resultant of P and Q, must not pass through the centre of the axis.

Suppose P to be on the point of preponderating. Draw the circular axis to a scale; make the angle CSR equal to



the limiting angle of resistance, on the same side of the centre as the preponderating force; measure the distances PM and MQ; then $PM.P = QM.Q$.

In this example the angle $CSR = 30°$; PM and MQ will be found from the same scale as the circle is measured to be 8.25 and 12.75 respectively.

$$\therefore 8.25\,P = 12.75\,Q;$$

$$\therefore P = \frac{12.75}{8.25}\,Q = 1.545\,Q.$$

Let $Q = 50$ lbs. then in the state bordering on motion $P = 1.545 \times 50 = 77.25$ lbs.

Neglecting friction $P.PC = Q.QC$.

$$P = \frac{Q.QC}{PC} = \frac{50 \times 12}{9} = \frac{600}{9} = 66.66 \text{ lbs.}$$

$\therefore 77.25 - 66.66 = 10.59$ lbs. due to friction in this case.

Had the weight been on the point of preponderating, then the limiting angle of resistance would have been measured on the opposite side of the centre towards Q.

Calculation.—

$$\text{since } \tan. CSM = f$$

$$\frac{\sin. CSM}{\cos. CSM} = f$$

$$\frac{\sin. CSM}{\sqrt{1 - \sin.^2 CSM}} = f$$

$$\sin.^2 CSM = f^2 - f^2 \sin.^2 CSM$$

$$\sin.^2 CSM + f^2 \sin.^2 CSM = f^2$$

$$\sin.^2 CSM = \frac{f^2}{1 + f^2}$$

$$\sin. CSM = \frac{f}{\sqrt{1 + f^2}}$$

$$CM = CS \sin. CSM = \frac{rf}{\sqrt{1 + f^2}}$$

$$PM = PC - CM = a - \frac{rf}{\sqrt{1+f^2}}$$

$$QM = QC + CM = b + \frac{rf}{\sqrt{1+f^2}}$$

$$P.PM = Q.QM$$

$$P\left(a - \frac{rf}{\sqrt{1+f^2}}\right) = Q\left(b + \frac{rf}{\sqrt{1+f^2}}\right)$$

$$Pa - P\frac{rf}{\sqrt{1+f^2}} = Qb + Q\frac{rf}{\sqrt{1+f^2}}$$

$$Pa = Qb + (P+Q)\frac{rf}{\sqrt{1+f^2}} \quad \ldots\ldots \quad (1)$$

where $a = PC$ and $b = QC$.

If M be on the other side of the centre C, we have

$$Pa = Qb - (P+Q).\frac{rf}{\sqrt{1+f^2}}.$$

When the directions of P and Q are not parallel, let them be produced till they meet, so that at the point of intersection they may make an angle θ; then the resultant instead of being $P+Q$ is by Art. (10),

$$\sqrt{P^2 + 2P.Q.\cos.\theta + Q^2},$$

The coefficient of friction f is so small, that for all practical purposes we may omit f^2; then equation (1) becomes

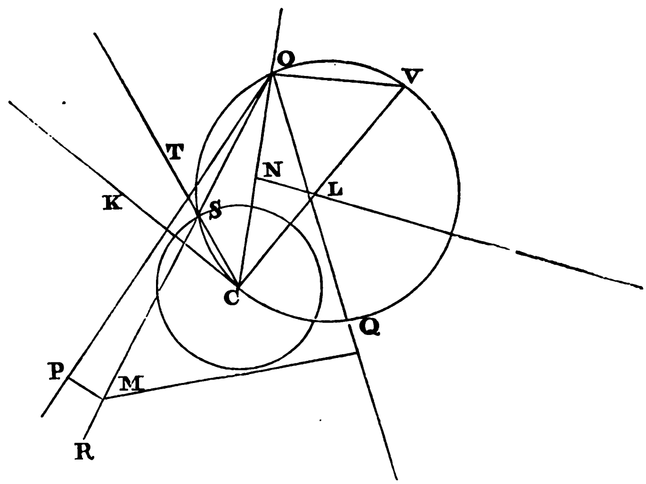
$$Pa = Qb + (P+Q)fr$$

$$\text{or } P = \frac{Qb + (P+Q)fr}{a - fr}$$

62. When the lever is movable about a cylindrical axis, and the directions of the forces are inclined at any angles, the relation between P and Q may be determined geometrically as follows:—

Set off the whole to a scale.

Let C be the centre of the axis, and O the point where the directions of the forces P and Q meet; join CO, bisect it in N, and draw NL perpendicular to it; make the angle OCK (on the side of the preponderating force P) equal to



the limiting angle of resistance; draw CL perpendicular to CK, intersecting NL in L, from L as a centre; describe the circle OSC, cutting the axis in S; join OS and produce it to R; then OR is the direction of the resultant in a state bordering on motion.

For since CK is perpendicular to CL, CK touches the circle, hence OCK is equal the angle OVC in the alternate segment; therefore $\angle\, OVC$ is equal to the limiting angle of resistance; but the angle in the alternate segment $+ \angle\, OSC =$ two right angles, since the quadrilateral $OSCV$ is incribed in a circle; and $\angle\, OSC + \angle\, OST =$ two right angles.

$\therefore$ angle in alternate segment $+ \angle\, OSC = \angle\, OSC + \angle\, OST$

Take away the common angle OSC, and we have

$OST =$ limiting angle of resistance, hence the resultant is in the direction OSR, when the lever is about to turn on

its axis. Hence in the state bordering on motion, the resistance is in the direction RS, and we have the three forces P, Q, and R in equilibrium. Take any point M in RS, and draw the perpendiculars MP and MQ, and measure them from the same scale, then $P \times MP = Q \times MQ$.

$$P = \frac{MQ}{MP} Q.$$

Or from O set off a distance $OQ =$ to the units of weight in Q, and draw from Q a line parallel to OP meeting the resultant in R suppose; then QR measured will give the units in the power P.

We have before observed, that the resultant of the forces must not pass through C; for then it would press perpendicularly through the middle of the axis, and there would be no more tendency to turn in the one direction than in the other; it is therefore evident that the resultant must act on that side of C, where the preponderating force acts, for motion to take place; and as the axis is on the point slipping on the bearance, the resultant must make with CS (figure page 71,) which is perpendicular to the surfaces in contact, an angle equal to the limiting angle of resistance.

Ex. 17. Three forces, which are to each other as 3, 4, 5, act upon a point and keep it at rest; required the angles at which these forces are inclined to each other.

Answer—126° 52′; 143° 8′; and 90°.

Ex. 18. Two forces, represented by 12lbs. and 15lbs. are inclined to each other at an angle of 60°; required the magnitude of the resultant, and its inclination to the greater.

Answer—Magnitude, 23.43; inclination to the greater force, 26° 20′.

Ex. 19. A weight W is sustained upon an inclined plane by three forces, each equal to $\frac{1}{3}$ W, one acting vertically upwards, another parallel to the plane, and the third horizontally; required the inclination of the plane.

Answer—57° 7′ 48″.

Ex. 26. A piece of timber, 24 feet long, being laid over a prop is found to balance itself, when the prop is 10 feet from the greater end, but removing the prop to the middle of the beam, it requires a man's weight of 200 lbs. standing on the less end, and also a weight of 20 lbs. at a distance of 4 feet from this end, to keep it in equilibrium; required the weight of the tree.

Answer—1280 lbs.

Ex. 27. A circular hoop is supported in a horizontal position, and three weights, 3, 4, 5 lbs. respectively are suspended over its circumference, by three strings meeting in the centre, what must be their position, that they may sustain each other?

Answer—One of the angles is a right angle.

Ex. 28. A cone of marble 20 feet high, and the diameter of its base 6 feet, is standing upright in a gentleman's garden, upon a horizontal plane; required the position of a rope with one end fixed at the vertex of this marble cone, so that it may require the least force or weight to be applied to the other end of the rope to overturn it.

Answer—The position is at right angles to the slant side.

Ex. 29. The arms of a bent lever are as 10 to 7, and they are inclined to each other at an angle of $112° 30'$; find a weight which, suspended at the end of the shorter arm, will balance 35 lbs. at the end of the longer arm, when the inclination of the longer arm to the horizon is twice as great as the inclination of the shorter.

Answer—$50 \sqrt{2 - \sqrt{2}}$ lbs.

Ex. 30. ACB is an arc of $120°$ in a vertical position, resting on a horizontal plane, with its concavity upwards; find its position when two weights of 8 and 12 lbs. suspended at the two extremities of the arc, keep it in equilibrium, (the arc being supposed to be without weight).

Answer—It rests on a point $79° 7'$ from the less weight.

Ex. 31. If G be the centre of gravity of a triangle, prove that the sum of the squares of the three sides of the

[illegible] and that [illegible]

[illegible] the weight of [illegible] [illegible], the

[illegible] 4 feet [illegible]

[illegible]

[illegible] ends on a horizontal plane,

[illegible] making an angle of 60° with the horizon:

[illegible] and the diameter of its

[illegible] the greatest globe

[illegible] without falling, when sus-

[illegible]

[illegible] horizontal way is an immovable flag-stone,

[illegible] which a carriage is to be drawn by the

[illegible] That part of the horse's breast to which

[illegible] feet high, and the distance

[illegible] what must be the diameter of

[illegible]

[illegible] whose length is 16, and base

[illegible] its vertex, its axis

[illegible] the horizon:

[illegible] quantity of the least

[illegible] base, that will just

[illegible]

[illegible] 6 feet in

[illegible] inches from

[illegible] to be at-

[illegible] the horizon;

[illegible]

[illegible] the three

[illegible] angled

[illegible]

Ex. 51. A cone of marble, the axis of which is 20 feet, and base diameter 6 feet, stands on the edge of its base the axis making an angle of 60° with the plane of the horizon; what must be the direction and quantity of the least power, applied to its vertex, that will just sustain the cone in that position?

Answer—1.37727 cwt.

Ex. 52. Required the solidity of the greatest cone, the diameter of its base being 6 feet, to just stand on an inclined plane, which makes an angle of 30° with the horizon; and find the highest point in its slant side where a weight may be placed without overturning it?

Answer—195.89.

Ex. 53. Find the centre of gravity of the frustum of a pyramid, a and b being the sides of the larger and smaller ends, and c the height.

Answer—Its distance from the larger end is

$$\frac{c}{4} \cdot \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}$$

Ex. 54. Give the calculation for Example 16, page 68.

Ex. 55. A given beam is suspended from a point in a vertical wall by a given string; find the position of equilibrium.

Ex. 56. Three equal and uniform rods are equally inclined to the vertical, their lower extremities being united in a fixed hinge, and their upper extremities by equal strings; find the tension of the strings.

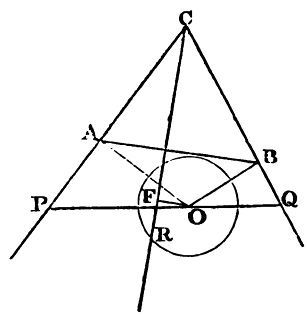
Ex. 57. Two equal and uniform rods, having their upper extremities connected by a hinge, rest on two pegs equally inclined to the horizon, the pegs being in the same horizontal plane; find the tension of the string connecting the extremities.

Ex. 58. A body in the form of an equilateral triangle, has a side coincident with an inclined plane on which it stands; find the least inclination of the plane, that the body may roll down.

$$\therefore\ P = \frac{MQ}{PM} \cdot Q = \frac{2.6}{2.34} \times 180 = 200 \text{lbs.}$$

Or set off $Oa = 180$, and drawing aR parallel to MQ, and meeting OM produced in R, then completing the parallelogram $aRbO$, and measuring aR or Ob from the same scale, it will be found $= 200$. Or, in order that the lines may not be too long, take them in fortieths, then Oa will be 4.5, that is, $\frac{1}{40}$ of 180, and aR or Ob will be found to be $= 5$, that is, $\frac{1}{40}$ of 200.

By Calculation, let POQ be the lever, O the centre of the axis, and let the directions of the forces make any given angles CPQ and CQP with the lever; then the angle PCQ will also be given. Suppose P to be on the point of preponderating. Then from O let fall the perpendiculars OA, OB, OF, on the directions of the two forces and their resultant. By the equality of moments



$$P \times OA = Q \times OB + R \times OF.$$

Join AB, and let $OA = a$, $OB = a_1$, $OF = a_2$, $CPQ = \alpha$, $CQP = \beta$, $AB = b$, $ACB = \phi$, radius of the axis $= r$, limiting angle of resistance $= \theta$, $PO = l$, $OQ = l_1$.

Then $$Pa = Qa_1 + Ra_2$$

$$Pa - Qa_1 = Ra_2$$

But (Art. 10) $$R^2 = P^2 + 2P.Q.\cos.\phi + Q^2$$

$$(Pa - Qa_1)^2 = a^2_2 (P^2 + 2P.Q.\cos.\phi + Q^2)$$

$$P^2a^2 - 2PQaa_1 + Q^2a^2_1 = a^2_2 (P^2 + 2P.Q.\cos.\phi + Q^2)$$

$$= a^2_2P^2 + 2a^2_2P.Q.\cos.\phi + a^2_2Q^2$$

$$P^2a^2 - a^2_2P^2 - 2P.Qaa_1 - 2a^2_2P.Q.\cos.\phi = a^2_2Q^2 - Q^2a^2_1$$

$$= -Q^2(a^2_1 - a^2_2)$$

Here $l=2, l_1=1, \sin.a=\sin.30=\frac{1}{2}, \sin.\beta=\sin.60=\frac{\sqrt{3}}{2}$, $l\sin.a=2\times\frac{1}{2}=1$, $l_1\sin.60=1\times\frac{\sqrt{3}}{2}=.866$, $r=\frac{1}{2}$, $\sin.\theta=.342$, $\cos.\phi=0$, $r^2\sin.\theta\cos.\theta=0$, $C^2=1.74996$, $r\sin.^2\theta\sin.^2\phi=.02924$.

Equation (3) becomes

$$P=\frac{1\times .866+.171\sqrt{1.74996-.02924}}{1-.02924}\times 180$$

$$=\frac{1.09}{.971}\times 180=201.$$

Ex. 60. A rough beam rests on a rough inclined and a rough horizontal plane, f being the co-efficient of friction, and i the angle of inclination of the plane; find the position of equilibrium.

Ex. 61. A round tower, 20 feet in diameter, and of the form of a cylinder surmounted by a cone, whose axis is to that of the cylinder as 1 to 8, is observed to have yielded 10 degrees from its perpendicular position, and it is known to have reached the exact limit of safety; required the height of the tower.

Answer—124.29 feet.

Ex. 62. Let radius of each pulley $=r$, weight of each pulley $=w$, and let there be n, such pulleys forming a system such as is represented in the figure page 41. Let P be the force which will raise a weight W by this system of pulleys. Prove that

$$P=\frac{W}{2^n}+\left(1-\frac{1}{2^n}\right)w.$$

Ex. 63. Two uniform beams of equal length are loosely connected at one extremity and placed across a given cylinder; find their position when they are in equilibrium, the pressure on the cylinder, and also the pressure at the

carpenter's rule, so that being freely suspended at one end, the lower arm when quiescent shall hang parallel to the horizon.

Answer—70° 31′ 43″.

Ex. 68. A flat board in the form of a square is supported upon two props with its plane vertical; determine its positions of equilibrium, friction being neglected, and the distance between the props being equal to half a side of the square.

THE INCLINED PLANE WITH FRICTION.

63. Let ϕ be the angle the power makes with the plane whose inclination is a, then, by Art. (46),

$P\cos.\phi \pm fR = W\sin.a$, by resolving the forces parallel to the plane.

And $P\sin.\phi + R = W\cos.a$, resolving perpendicular to the plane.

Eliminating R the reaction of the plane, and we have

$$\frac{P}{W} = \frac{\sin.a \mp f\cos.a}{\cos.\phi \mp f\sin.\phi}.$$

The upper sign to be taken when the friction acts up the plane, and the lower when it acts down.

$$\text{but } f = \tan.\theta = \frac{\sin.\theta}{\cos.\theta}$$

$$\frac{P}{W} = \frac{\sin.a \mp \dfrac{\sin.\theta}{\cos.\theta}\cos.a}{\cos.\phi \mp \dfrac{\sin.\theta}{\cos.\theta}\sin.\phi}$$

$$= \frac{\sin.a\cos.\theta \mp \sin.\theta\cos.a}{\cos.\theta\cos.\phi \mp \sin.\theta\sin.\phi}$$

$$= \frac{\sin.(a \mp \theta)}{\cos.(\phi \pm \theta)}$$

If friction acts down the plane we have

$$P = \frac{\sin.(a+\theta)}{\cos.(\phi-\theta)} W \ldots (1)$$

Now, if we wish to find the direction of the least traction, we must make P the least possible; but P is the least possible when the denominator is the greatest possible, that is, when $\phi - \theta = 0$, and therefore

$$\phi = \theta \ldots\ldots\ldots (2)$$

The best direction is, therefore, when the power makes an angle with the plane equal to the limiting angle of resistance.

If $a = 0$, the plane becomes horizontal;

$$\therefore P = \frac{\sin.\theta}{\cos.(\phi-\theta)} \ldots\ldots (3)$$

and, for the least traction on a horizontal plane the direction must make with that plane an angle equal to the limiting angle of resistance.

At Art. (46), Cor. 2, it was shown that the least power necessary to sustain a body on an inclined plane, was whe[illegible] the power acted parallel to the plane, but in that ca[illegible] friction did not enter into the calculation; when fricti[illegible] is considered the above results hold good: it is necess[illegible] however, to remark, that the limiting angle of resis[illegible] will vary according to the nature of the road; the r[illegible] the road the greater that angle will be; hence, it [illegible] that when a horse draws a load along a plane, [illegible] should never be parallel to that plane. [illegible] the road the nearer the direction m[illegible] parallelism. On railways the line o[illegible] nearly parallel to the road, the frictio[illegible] small.

If the bodies are equal, we have,

$$M : m :: V : v$$

If the bodies move with equal velocities, then,

$$M : m :: W : w$$

If two bodies move with velocities which are inversely as their quantities of matter, their momenta will be equal. For then,

$$M : m :: v : V.$$

And since the product of the extreme terms of a proportion is equal to the product of the mean terms, we have

$$MV = mv$$

If a body, the weight of which is 8lbs. moves with a velocity of 10 feet per second; and another body, the weight of which is 10lbs. moves with a velocity of 5 feet per second; then,

$$M : m :: 8 \times 10 : 10 \times 5 :: 8 : 5.$$

That is, the momentum of the former is to the momentum of the latter as 8 to 5.

Or, if we take the momentum of the former to that of the latter as 8 to 5, and their velocities as 2 to 1, then their weights are to each other as $\frac{8}{2}$ to $\frac{5}{1}$, that is, as 4 to 5.

And if the momenta of four bodies are as 1, 2, 3, 4, and their weights as 3, 4, 5, 6, then their velocities are as $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}$; or as $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}$; or, which is the same, 10, 15, 18, 20.

The battering-ram of Vespasian weighed, suppose, 100,000lbs.; and was moved, let us admit, with such a velocity, by strength of hands, as to pass through 20 feet in one second of time; and this was sufficient to demolish

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] as the [illegible]

[illegible] which [illegible] a [illegible]

For, in any given time, the momentum which is generated will be proportional to the force which generates it. And since the force has the same efficacy in each instant of time, the whole momentum will be as the sum of these instants, or whole time. Consequently, the whole momentum, or quantity of motion generated, is in the compound ratio of the force and time of acting.

Cor. 1. The quantity of motion lost or destroyed in any time, by a force acting in an opposite direction, is also in the compound ratio of the force and time.

Cor. 2. The velocity generated or destroyed in any time, is as the force and time directly, and quantity of matter reciprocally; that is, v is as $\frac{ft}{b}$; where v denotes the velocity, f the force, and b the body, or quantity of matter. For, by Art. (70), the momentum is as the quantity of matter and velocity; therefore, the velocity is as the momentum directly and quantity of matter reciprocally; that is, by this Art., as the force and time directly and quantity of matter reciprocally.

Cor. 3. Hence, if the body be given, the velocity will be in the compound ratio of the force and time; and if the force be given, the time is in the compound ratio of the quantity of matter and the velocity, or as the momentum.

74. If a given body be urged by a constant and uniform force, the space which is described by the body from the beginning of the motion is as the force and square of the time.

For, suppose the time to be divided into an indefinite number of equal parts. Then, in each of these equal parts of time, the space described will be as the velocity gained; that is, by Art. (73), Cor. 3, as the force and time from the beginning. And the sum of all the spaces, or the whole space described, will be as the force and the sum of all the equal parts of time from the beginning. If we put $t=$ the whole time, the whole space described

[illegible]

[illegible]

[illegible] sec. [illegible]

[illegible] sec [illegible]

[illegible]

[illegible]

gravity,* then the space described is as the square of the velocity. The space is also as the square of the time. Hence the velocity is as the time.

We may now show the application of these proportions to falling bodies.

ON GRAVITY.

76. Gravity is that power or force which causes bodies to approach each other. This universal principle, which pervades the whole system of nature, may be enunciated as follows.

The mutual tendency of two bodies towards each other increases in the same proportion as their masses are increased, and the square of their distance is decreased; and it decreases in proportion as their masses are decreased, and as the square of their distance is increased.

Thus, if any two bodies *A* and *B* be placed in free space at a given distance from each other, and if we suppose the mass of *A* to be double the mass of *B*, then *B* will move towards *A* with double the velocity with which *A* moves towards *B*.

Also, if at any given distance *A* tends towards *B* with a given force, at double that distance *A* will tend towards *B* with one-fourth of that force, at treble the distance with one-ninth of that force, and so on.

Philosophers have formed theories of various kinds to account for this universal principle. Some have considered that it is produced by particles emanating from a centre or centres. But, as Dr. Paley† very justly observes, we are totally at a loss to comprehend how particles streaming from a centre can possibly draw bodies toward

* The weights of all bodies in the same place are proportional to the quantities of matter they contain, without any regard to their bulk, figure, or kind. For twice the matter will be twice as heavy, thrice the matter thrice as heavy, and so on.

† We would particularly recommend to our readers the perusal of *Paley's Natural Theology*, it being a work replete with useful information.

that centre. The impulse, if impulse there be, is all the other way.

We are equally at a loss if we consider that the effect is produced by a conflux of particles flowing towards a centre, and carrying down all bodies along with it; for, if such a fluid exists, it must act very powerfully, and at the same time offer no resistance whatever to bodies moving in it, which is contrary to the known constitution of fluids.

We are also utterly unable to conceive how one body can act upon another at a distance, or, in other words, that a body can act where it is not; and it appears no more ridiculous to assert that a body can act when it ceases to exist, than to assert that a body can act where it does not exist.

Indeed, every hypothesis relating to the cause of gravity is embarrassed with insuperable difficulties. It seems, as it were, to be among the arcana of the Almighty; and, until a theory can be advanced which can clear up all these difficulties, it is more prudent to conclude, with the illustrious Newton, "that in the absence of the secondary cause of gravity, we may attribute it to the final cause of all things, the finger of God, the constant impression of divine power."

TERRESTRIAL GRAVITY.

77. Terrestrial Gravity is that force by which bodies are urged towards the centre of the earth, and it is measured by the velocity generated in a second of time. As has been already remarked, experiments show that a falling body describes $16\frac{1}{12}$ feet in the first second, and it has then acquired a velocity of $32\frac{1}{6}$ feet, which is therefore the true measure of the force of gravity.

There appears to be an inequality of the action of gravity upon different kinds of matter near the surface of the earth. But this arises entirely from the resistance which they meet with in passing through the air; for, in the

exhausted receiver of an air-pump, all bodies fall equally; a guinea acquires no greater velocity than a feather; an ounce of feathers and a ton of gold fall through equal spaces in equal times, and will reach the bottom of the receiver exactly at the same instant of time.

78. Since it is known by experiment that the space which a body describes by falling freely from rest is $16\frac{1}{12}$ feet in the first second of its fall, by Art. (74), Cor. 1, the body will have acquired a velocity of $32\frac{1}{6}$ feet; that is, the body will have acquired a velocity which, if continued uniformly, would carry it through twice the space in the same time, if gravity ceased at that instant to act.

Put s for the space described by the body in any other time t, and v the velocity acquired; and since the spaces are as the squares of the velocities,

$$s : 16\tfrac{1}{12} :: v^2 : (32\tfrac{1}{6})^2 = 4(16\tfrac{1}{12})^2;\text{*}$$

$$\therefore\ s = \frac{16\frac{1}{12}v^2}{4(16\frac{1}{12})^2} = \frac{v^2}{64\frac{1}{3}};\ \therefore\ v = 2\sqrt{16\tfrac{1}{12}s}.$$

The spaces are also as the squares of the times.

$$s : 16\tfrac{1}{12} :: t^2 : 1^2;$$

$$\therefore\ s = 16\tfrac{1}{12}t^2;\ \therefore\ t = \sqrt{\frac{s}{16\frac{1}{12}}}.$$

Also, the velocities are as the times; that is, the velocity varies as the time varies.

$$t : 1 :: v : 32\tfrac{1}{6};$$

$$\therefore\ v = 32\tfrac{1}{6}t;\ \text{and}\ t = \frac{v}{32\frac{1}{6}}.$$

Collecting these:

$$s = \frac{v^2}{64\frac{1}{3}} = 16\tfrac{1}{12}t^2 = \frac{tv}{2};$$

* The square of any quantity is equal to four times the square of half that quantity; thus $16\frac{1}{12}$ is half of $32\frac{1}{6}$; hence $4\ (16\frac{1}{12})^2 = (32\frac{1}{6})^2$.

$\frac{1}{12}$, and the product will give the space fallen through feet.

Example.—Through what space will a body fall in 10 onds?

$$10^2 = 100, \text{ and } 100 \times 16\tfrac{1}{12} = 1608\tfrac{1}{3} \text{ feet.}$$

By the formula,

$= 16\frac{1}{12} \times t^2 = 16\frac{1}{12} \times 10^2 = 1608\frac{1}{3}$, the same as before.

en the space which a heavy body has fallen through, to find the velocity acquired.

ule 3.—Multiply the space fallen through in feet by , and twice the square root of the product will give velocity acquired in feet.

xample.—What velocity will a body acquire in falling feet?

$00 \times 16\frac{1}{12} = 1608.3$, the square root of which is 40.104, $40.104 \times 2 = 80.208$ feet per second.

By formula,

$= 2\sqrt{16\frac{1}{12}s} = 2\sqrt{16\frac{1}{12} \times 100} = 2\sqrt{1608.3} = 80.208$ second.

en the time which a heavy body has been in falling, to find the velocity it has acquired.

ule 4.—Multiply the time of falling in seconds by $32\frac{1}{6}$, this product will give the velocity acquired in feet.

xample.—What velocity will a body acquire by falling seconds?

$$10 \times 32\tfrac{1}{6} = 321\tfrac{2}{3} \text{ feet, the velocity per second.}$$

By formula,

$$v = 32\tfrac{1}{6}t = 10 \times 32\tfrac{1}{6} = 321\tfrac{2}{3}, \text{ as before.}$$

find the time which a heavy body will be in falling through a given space.

ule 5.—Divide the space in feet by $16\frac{1}{12}$, and the

$$v = 2\sqrt{16\tfrac{1}{12}s} = 32\tfrac{1}{6}t = \frac{2s}{t};$$

$$t = \sqrt{\frac{s}{16\frac{1}{12}}} = \frac{v}{32\frac{1}{6}} = \frac{2s}{v}.$$

79. Also, since the spaces described by falling bodies are as the squares of the times, if those times be represented by the numbers, 1, 2, 3, 4, &c. the spaces described in those times will be as 1, 4, 9, 16, &c. which are the squares of 1, 2, 3, 4, &c. respectively; and the spaces described in a series of equal portions of time will be as the odd numbers, 1, 3, 5, 7, &c.; that is, if a body fall through $16\frac{1}{12}$ feet in the first second, it will fall through $3 \times 16\frac{1}{12}$ in the next second, $5 \times 16\frac{1}{12}$ in the third, and so on.

Also, the velocities are as the numbers, 1, 2, 3, 4, &c. since the velocities are as the times.

The above may be expressed in the form of rules, for the use of those who do not understand the application of formulæ, in the following manner:

Given the velocity which a heavy body acquires in falling, to find the space it has fallen through to acquire that velocity.

Rule 1.—Divide the square of the acquired velocity by $64\frac{1}{3}$, and the quotient will give the space in feet which the body has fallen through to acquire that velocity.

Example.—How far must a body fall to acquire a velocity of 120 feet per second?

$120^2 = 14400$, and $14400 \div 64\frac{1}{3} = 223.8$ feet.

By the formula,

$$s = \frac{v^2}{64\frac{1}{3}} = \frac{120^2}{64\frac{1}{3}}$$

Given the time a heavy

sp

Rule 2.—Multi

But if the body is projected upwards, then the distance of the body from the point of projection is $t \times v - 16\frac{1}{12} \times t^2$; for gravity, in this case, acts in an opposite direction to the motion of the body. Hence we have the following rules:—

Case 1.—When a body is projected downwards with a given velocity, multiply the square of the time in seconds by $16\frac{1}{12}$, and the velocity of projection in feet by the number of seconds the body is in motion; then the sum of these products will give the space moved through by the body.

Case 2.—If the body is projected upwards, then the difference of the above products will give the distance of the body from the point of projection.

Ex. 1.—If a body be projected downwards with a velocity of 20 feet per second, through what space will it fall in 6 seconds?

Now, $6^2 = 36$, and $16\frac{1}{12} \times 36 = 579$ feet, the space passed through by the action of gravity.

Then, $20 \times 6 = 120$ feet, the space passed through by the uniform impulse.

Hence the whole space is $579 + 120 = 699$ feet.

By the formula,

Here, $v = 20$ and $t = 6$; $\therefore\ t \times v + 16\frac{1}{12} \times t^2 = 6 \times 20 + 16\frac{1}{12} \times 6^2 = 699$ feet.

Ex. 2.—If a body is projected upwards with the velocity of 250 feet per second, how far will it rise in 2 seconds?

$250 \times 2 = 500$ feet, the space which the body would pass over if gravity did not act.

Then, $2^2 = 4$, and $16\frac{1}{12} \times 4 = 64\frac{1}{3}$, the retardation arising from gravity.

Hence, $500 - 64\frac{1}{3} = 435\frac{2}{3}$ feet.

By the formula,

Here, $v = 250$, and $t = 2$; $\therefore\ t \times v - 16\frac{1}{12} \times t^2 = 2 \times 250 - 16\frac{1}{12} \times 2^2 = 435\frac{2}{3}$ feet.

Ex. 3.—If a body be projected upwards with a velocity of 30 feet per second, through what space will it ascend before it begins to return?

It is evident, by Art. (80), that if a body be projected upwards with a given velocity, it will ascend to the same height from which it must fall to acquire that velocity; therefore Rule 1 applies to this problem; that is, divide the square of the velocity of projection in feet by $64\frac{1}{3}$, and the quotient will give the height to which the body will ascend.

$30^2 = 900$ feet, and $900 \div 64\frac{1}{3} = 14$ feet nearly.

Ex. 4.—If a body be projected vertically upwards with a velocity of 100 feet per second, it is required to find the place of the body at the end of 10 seconds.

By the Rule, $100 \times 10 = 1000$ feet, the space which a body would move through if gravity did not act; and $16\frac{1}{12} \times 10^2 = 1608\frac{1}{3}$, the retardation arising from gravity.

Hence, $1000 - 1608\frac{1}{3} = -608\frac{1}{3}$ feet, the negative sign shows that the body will be $608\frac{1}{3}$ feet below the point of projection.

ON THE MOTION OF BODIES ON INCLINED PLANES.

82. In treating of the equilibrium of an inclined plane, it was shown that the force on an inclined plane bears the same proportion to the force of gravity as the height of the plane bears to its length: that is, the force which accelerates the motion of a body down an inclined plane, is that fractional part of the force of gravity which is represented by the height of the plane divided by its length. Therefore, if h represents the height of the plane, and l its length, then $\frac{h}{l}$ will represent the accelerating force. In the formulæ, page 98, for f put $\frac{h}{l}$; or, which is the same, substitute sin. i for f, i being the angle of inclination; and we have,

$$s = \tfrac{1}{2}tv = \frac{16\frac{1}{12}ht^2}{l} = \frac{lv^2}{64\frac{1}{3}h}$$

$$v = \frac{2s}{t} = \frac{32\frac{1}{6}ht}{l} = \sqrt{\frac{64\frac{1}{3}.hs}{l}}$$

$$t = \frac{2s}{v} = \frac{lv}{32\frac{1}{6}h} = \sqrt{\frac{ls}{16\frac{1}{12}h}}$$

$$\frac{h}{l} \text{ or sin. } i = \frac{v}{32\frac{1}{6}t} = \frac{s}{16\frac{1}{12}t^2} = \frac{v^2}{64\frac{1}{3}.s}$$

$$s = \sin. i \times 16\tfrac{1}{12}t^2 = \frac{v^2}{64\frac{1}{3}\sin. i}$$

$$v = \sin. i \times 32\tfrac{1}{6}t = \sqrt{\sin. i \times 64\tfrac{1}{3}s}$$

$$t = \frac{v}{32\frac{1}{6}\sin. i} = \sqrt{\frac{s}{16\frac{1}{12}\sin. i}}$$

Given the length and height of an inclined plane, to find the space which a body will move through in a given time.

Rule 1.—Multiply the height of the plane in feet by the square of the given time in seconds, and divide this product by the length of the plane, also in feet; and this quotient, multiplied by $16\frac{1}{12}$, will give the space in feet descended or ascended.

Example.—The length of an inclined plane is 100 feet, and the height 50 feet; what space will a body descend through in 3 seconds?

$50 \times 3^2 = 450$, then $450 \div 100 = 4\frac{1}{2}$, and $16\frac{1}{12} \times 4\frac{1}{2} = 72.07$ feet, the space required.

By formula,

$$s = \frac{16\frac{1}{12} \times ht^2}{l} = \frac{16\frac{1}{12} \times 50 \times 3^2}{100} = 72.07 \text{ feet.}$$

Given the length and height of an inclined plane, to find the space which a body must move through to acquire a given velocity.

Rule 2.—Multiply the length of the plane in feet by the square of the acquired velocity; and this product, divided

will give

along the

per second?

feet,

feet, the space

inclined plane, to find given time.

height of the plane in product, divided by will give the velocity ac-

height of the inclined plane will a body acquire in 2

$100 = 32$ feet per

feet, the same as before.

considerations, for force on the plane gravity: it will there- it will generate in the force of gravity

plane, to find in descending

of the plane by

the space described, all in feet; and if this product be divided by the length of the plane, also in feet, the square root of the quotient will give the velocity acquired in feet.

Example.—What velocity will a body acquire in descending down an inclined plane, the length of which is 20 feet and height 1 foot.

$64\frac{1}{3} \times 1 = 64\frac{1}{3}$, then $64\frac{1}{3} \times 20 = 643\frac{1}{3}$, and $643\frac{1}{3} \div 20 = 64\frac{1}{3}$, the square root of which is 8.02 nearly, the acquired velocity.

By formula,

$$v = \sqrt{\frac{64\frac{1}{3} \times hs}{l}} = \sqrt{\frac{64\frac{1}{3} \times 1 \times 20}{20}} = \sqrt{64\frac{1}{3}} = 8.02$$

feet, as before.

Given the length and height of an inclined plane, to find the time in which a body will acquire a given velocity.

Rule 5.—Multiply the length of the plane in feet by the given velocity in feet; and this product, divided by $32\frac{1}{6}$ times the height in feet, will give the time in seconds.

Example.—If the height of an inclined plane be 1 foot, and the length 40 feet, in what time will a body descending down this plane by the force of gravity acquire a velocity of 5 feet per second?

$40 \times 5 = 200$, and $200 \div 32\frac{1}{6} \times 1 = 6.22$ seconds nearly.

By the formula,

$$t = \frac{lv}{32\frac{1}{6}h} = \frac{40 \times 5}{32\frac{1}{6} \times 1} = 6.22 \text{ seconds.}$$

Given the length and height of an inclined plane, to find the time in which a body will describe a given space.

Rule 6.—Multiply the length of the plane in feet by the space also in feet; and if this product be divided by $16\frac{1}{12}$ times the height of the plane in feet, the square root of the quotient will give the time in seconds.

Example.—Let the length and height of the plane be the same as in the last Example, to find how long a body will be in descending down this plane.

Here the length of the plane and the given space are the same; therefore $40 \times 40 = 1600$, then $1600 \div 16\frac{1}{12} \times 1 = 100$ nearly, the square root of which is 10 seconds nearly.

Given the time which a heavy body is descending down an inclined plane, and the velocity acquired, to find the angle of inclination of the plane.

Rule 7.—Divide the acquired velocity by $32\frac{1}{6}$ times the time in seconds, and this quotient will give the sine of the angle of inclination.

Example.—If a body which moves down an inclined plane for 3 seconds acquires a velocity of $48\frac{1}{4}$ feet per second, what is the inclination of the plane?

$48\frac{1}{4} \div 32\frac{1}{6} \times 3 = \dfrac{48\frac{1}{4}}{96\frac{1}{2}} = \frac{1}{2}$, which is the sine of 30°.

By formula,

$$\sin. i = \frac{48\frac{1}{4}}{32\frac{1}{6} \times 3} = \frac{48\frac{1}{4}}{96\frac{1}{2}} = \tfrac{1}{2}, \text{ the same as before.}$$

Also, this shows that the length of the plane is twice its height, or the accelerating force on the plane is only half the accelerating force of gravity.

To find the angle of inclination when the space described and time are given.

Rule 8.—Divide the space described in feet by $16\frac{1}{12}$ times the square of the time, and the quotient will give the sine of the angle of inclination.

Example.—A body descends $21\frac{4}{9}$ feet from rest along an inclined plane in 2 seconds; required the inclination of the plane.

$21\frac{4}{9} \div 16\frac{1}{12} \times 2^2 = \dfrac{21\frac{4}{9}}{16\frac{1}{12} \times 4} = \dfrac{21\frac{4}{9}}{64\frac{1}{3}} = \frac{1}{3}$; that is, the length of the plane is three times its height.

By formula,

$$\sin. i = \frac{s}{16\frac{1}{12}t^2} = \frac{21\frac{4}{9}}{16\frac{1}{12} \times 2^2} = \frac{21\frac{4}{9}}{64\frac{1}{3}} = \tfrac{1}{3}.$$

To find the angle of inclination, when the space described and the velocity acquired are given.

Rule 9.—Divide the square of the acquired velocity by $64\frac{1}{3}$ times the space in feet, and the quotient will give the sine of the angle of inclination.

Example.—A body, by descending 100 feet along an inclined plane, acquires a velocity of $64\frac{1}{3}$ feet; what proportion does the accelerating force on the plane bear to the accelerating force of gravity?

$$(64\tfrac{1}{3})^2 \div 64\tfrac{1}{3} \times 100 = \frac{(64\frac{1}{3})^2}{64\frac{1}{3} \times 100} = \frac{64\frac{1}{3}}{100} = \frac{193}{3 \times 100} = \frac{193}{300}$$

Note.—The acquired velocity is always given in feet per second.

If the proportion which the length of the plane bears to the height be given, we must substitute these proportions in the foregoing rules.

Thus, suppose that the length of a plane is to its height as 2 to 1, then in these rules use 2 for the length of the plane, and 1 for the height, and the conclusions will be equally as true as those where the length and height are absolutely given.

Example.—In what time will a body descend 30 feet down an inclined plane which rises 1 foot in 10?

Here the length is to the height as 10 to 1; and using 10 for the length and 1 for the height, in Rule 6, we have, $10 \times 30 = 300$, and $300 \div 16\frac{1}{12} \times 1 = 18.65$, the square root of which is 4.32 seconds nearly.

Or thus, by the formula,

$$t = \sqrt{\frac{ls}{16\frac{1}{12}h}} = \sqrt{\frac{10 \times 30}{16\frac{1}{12} \times 1}} = 4.32 \text{ seconds.}$$

83. If a body be projected down an inclined plane with a given velocity, then the distance which the body will be from the point of projection in a given time will be $t \times v + \frac{h}{l} \times 16\frac{1}{12}t^2$; but if the body be projected upwards, then the distance of the body from the point of projection will

be $t \times c$ [illegible] $\frac{h}{l} \times 16\frac{1}{12}t^2$ retaining the same notation as before.

Ex. 1. If a body be projected with a velocity of 40 feet per second down an inclined plane the length of which is [illegible] times its height what space will it move through in 6 seconds?

Here $t =$ [illegible] $c = 40$ and $\frac{h}{l} =$ [illegible].

Hence [illegible] $\frac{h}{l} \times 16\frac{1}{12}t^2 =$ [illegible] $\times 40 +$ [illegible] $\times 16\frac{1}{12} \times 6^2 = 433$ [illegible]

[illegible] be projected upwards with a velocity [illegible] what distance will it be from t[illegible] [illegible] 3 seconds?

[illegible] 40 and $\frac{h}{l} =$ [illegible].

[illegible] $\times 16\frac{1}{12}t^2 = 40 \times 3 -$ [illegible] $\times 16\frac{1}{12} \times 3^2 = 71$ [illegible]

INERTIA.

[illegible] the passiveness of matter; that [illegible] power of putting itself into motion, [illegible] of stopping itself when put into [illegible] of an external force, for it requires [illegible] to stop a body as it requires to put it in [illegible]

[illegible] of motion, arising from inertia, is always [illegible] the mass of the body; for the inertia of a [illegible] composed of the inertia of all its parts; [illegible] body contain twice as much matter as [illegible] the communication of motion twice [illegible] properly speaking, the former body [illegible] much force to move it with a given [illegible] will require.

[illegible] that the momentum of a body [illegible] velocity and quantity of matter;

consequently, the velocity is proportional to the momentum divided by the quantity of matter.

The force which generates momentum in a body is called a moving force, and the force which generates velocity is called an accelerating force; therefore, if we substitute the moving force, which is proportional to the momentum, for the momentum, and the accelerating force, which is proportional to the velocity, for the velocity, we have $f = \frac{P}{M}$; P being = the moving force or pressure, f = the accelerating force, and M = the mass or quantity of matter.

If any two bodies P and W be suspended over a pulley moveable about an axis, then, if P be heavier than W, we have $P - W$ for the moving force, but $P + W$ is the mass moved; hence

$\frac{P-W}{P+W}$ will represent the accelerating force.

But if the inertia of the pulley be taken into consideration, which call I, then the accelerating force is $\frac{P - W}{P + W + I}$.

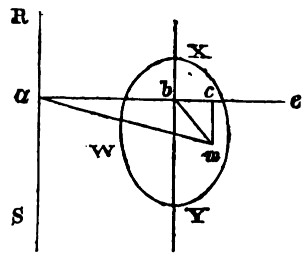
If, in the formulæ, page 98, we substitute $\frac{P-W}{P+W}$ for f, the theorems will apply to this case.

Ex. 1.—If a weight of 6 lb. act upon a weight of 4 lbs. over a pulley, what space will it descend in 6 seconds?

$$\text{Here, } \frac{P-W}{P+W} = \frac{6-4}{6+4} = \frac{2}{10} = \frac{1}{5} = f;$$

$$s = 16\tfrac{1}{12}ft^2 = 16\tfrac{1}{12} \times \frac{1}{5} \times 36 = 115\tfrac{4}{5}.$$

Ex. 2.—If a weight of 6 lbs. be attached to a weight of 2 lbs. by means of a cord going over a pulley, how far will the heavier weight descend in 6 seconds?



To find the moment of inertia with respect to any axis whatever. Let RS be any axis whatever, and XY an axis passing through the centre of gravity parallel to it; m, any particle of the body WXY. Draw any line ae from m, let fall the perpendicular mc, and join mb; let $mb = r_{,}$, ab, the distance between the axes $= k$, and $am = r$; then, by Prop. 12, second book of Euclid, we have

$$(am)^2 = (ab)^2 + (mb)^2 + 2ab \,.\, bc;$$
$$r^2 = k^2 + r_{,}^2 + 2k \,.\, d.$$

Multiplying by the mass m we have

$mr^2 = mk^2 + mr_{,}^2 + 2k \,.\, m \,.\, d;$

$m'r'^2 = m' \,.\, k^2 + m'r_{,}'^2 + 2k \,.\, m' \,.\, d'$, for mass m';

$m''r''^2 = m''k^2 + m''r_{,}''^2 + 2k \,.\, m'' \,.\, d''$, for mass m''.

&c.

By addition we have

$$mr^2 + m'r'^2 + m''r''^2 + \&c. = (m + m' + m'' + \&c.)\, k^2 + mr_{,}^2 + m'r_{,}'^2 + m''r_{,}''^2 + \&c.; + 2k\,(md + m'd' + m''d'' + \&c.)$$

Now, $mr^2 + m'r'^2 + \&c.$ = moment of inertia with respect to the axis RS, which call I; and $mr_{,}^2 + m'r_{,}'^2 + \&c.$ = moment of inertia with respect to the axis XY; $m + m' + m'' + \&c.$ is the mass of the whole body; and, by the property of the centre of gravity, $md + m'd' + m''d'' + \&c. = 0$.

Hence, if $I_{,}$ be the moment of inertia with respect to the axis XY, through the centre of gravity, and M be the mass of the whole body, we have

$$I = I_{,} + Mk^2.$$

Hence, the moment of inertia of a body about an axis which does not pass through the centre of gravity, is equal

[illegible] the moment of inertia of the same body about a parallel axis passing through the centre of gravity, together with the product of the mass of the body, and the square of the distance between the two axes.

If any system of bodies revolve round a fixed axis, a point may be found, such that if the whole mass of the system were collected in it, the moment of inertia would be the same as before; that is, the sum of the products of each body, into the square of its distance from the axis of motion, is equal to the sum of all the bodies, multiplied by the square of the distance of this point from the axis of motion.

Or, in a single body, a point may be found such that if the whole mass of the body were collected in this point, the moment of inertia would be the same as before. This point is called the centre of gyration, and its distance from the axis of rotation is called the radius of gyration.

Or the centre of gyration may be defined to be that point into which, if the whole mass be collected, the same angular velocity will be generated in the same time, by a given [illegible] any place, as in the body or system itself.

[illegible] motion of a body, or system of bodies, is the [illegible] connecting any point and the axis of motion.

[illegible] weight, or power, which acts at the distance [illegible] motion, to give rotation to a body or [illegible] of which is W, and the distance of the [illegible] from the axis of motion $= k$, then the [illegible]

$$\frac{Pr^2}{Pr^2 + Wk^2}.$$

[illegible] be put in motion by a power [illegible] then the inertia of P is nothing;

$$\frac{Pr}{Wk^2}.$$

[illegible] gyration, in some of the [illegible]

In a circular wheel of uniform thickness . . $k = \frac{1}{2} r \sqrt{2}$
In the circumference of a circle revolving about diameter . . . . $k = \frac{1}{2} r \sqrt{2}$
In the plane of a circle ditto . . . . . $k = \frac{1}{2} r$
In a solid sphere ditto . . . . $k = r \sqrt{\frac{2}{5}}$
In a circular ring, the radii of which are R and r revolving about the centre . $k = \sqrt{\left(\frac{R^2 + r^2}{2}\right)}$ *
In a cone revolving about its vertex $k = \frac{1}{2} \sqrt{(\frac{12}{5} a^2 + \frac{3}{5} r^2)}$
In a cone revolving about its axis . $k = r \sqrt{\frac{3}{10}}$
In a straight lever, the arms of which are R and r . . . . $k = \sqrt{\left(\frac{R^3 + r^3}{3(R + r)}\right)}$
In a paraboloid, the radius of the base of which is R . . . . $k = R \sqrt{\frac{1}{3}}$

Ex. 1.—A cylinder, the weight of which is 80 lbs., is put in motion by a weight of 20 lbs. attached to a string which is coiled round the cylinder. How far will the weight descend in 6 seconds?

The accelerating force $f = \dfrac{Pr^2}{Pr^2 + Wk^2}$.

Here, $P = 20$ lbs., $W = 80$ lbs., and $k = \frac{1}{2} r \sqrt{2}$;

$$\therefore \frac{Pr^2}{Pr^2 + Wk^2} = \frac{Pr^2}{Pr^2 + W \times \frac{1}{2} r^2} = \frac{P}{P + \frac{1}{2} W}$$
$$= \frac{20}{20 + 40} = \frac{1}{3}.$$

That is, the accelerating force is $\frac{1}{3}$ of the accelerating force of gravity; and by formula, page 98,

$$s = 16\tfrac{1}{12} f t^2 = 16\tfrac{1}{12} \times \frac{1}{3} \times 6^2 = 193 \text{ feet.}$$

* In Dr. Gregory's Mathematics for Practical Men, this is given, $\sqrt{\left(\frac{R^4 - r^4}{2R^2 - 2r^2}\right)}$; but it may be reduced so as to agree with the above, as follows:—

$$\sqrt{\left(\frac{R^4 - r^4}{2R^2 - 2r^2}\right)} = \sqrt{\left\{\frac{(R^2 + r^2) \times (R^2 - r^2)}{2(R^2 - r^2)}\right\}} = \sqrt{\left(\frac{R^2 + r^2}{2}\right)}$$

more bodies, or of one body from the figure and extent of which we are not permitted to abstract.

89. The centre of oscillation of a compound pendulum is a point in it at such a distance from the centre of suspension, that a simple pendulum, of a length equal to that distance, will have the same angular velocity with the compound pendulum itself.

90. The centre of percussion, which is generally in the same point as the centre of oscillation, may be explained as follows:—

In striking any body with a bar or lever, it is always found that if the blow is given at or near the end of the bar, it will jar, or attempt to fly out of the hand; and if the blow is given by that part of the bar near the hand, it will also jar, and attempt to fly from it. Now there evidently must be a point between these two, where, if a stroke is given, the full effect of the blow will be sensible, and the bar will remain at rest, without jarring the hand. This point is called the centre of percussion, or the point in a striking body where, if it strike another, the effect will be most powerful; and, as the centre of gravity of a body is a point on which, if suspended, the body would be in equilibrio, so the centre of percussion is a point in which the whole momentum of the moving body is placed to produce the greatest effect.

ON THE SIMPLE PENDULUM.

91. It has been found, by many very accurate experiments, that a pendulum which vibrates seconds in the latitude of London is $39\frac{1}{8}$ inches in length. This being known, we can find the length of a pendulum which will make any number of vibrations in a given time, as follows:

Bring the given time into seconds; then, as the square of the number of vibrations given, is to the square of the given number of seconds, so is $39\frac{1}{8}$, to the length of the required pendulum in inches.

Example.—What must be the length of a pendulum, so as to make 80 vibrations in a minute?

Here the given time is 60 seconds.

$$6400 : 3600 :: 39\tfrac{1}{8} : \frac{39\frac{1}{8} \times 3600}{6400} = \frac{140850}{6400} = 22 \text{ inches.}$$

Therefore, if the length of a pendulum be required, so as to make a given number of vibrations in a minute, divide 140850 by the square of the number of vibrations given, and the quotient will be the length of the pendulum.

Example.—What must be the length of a pendulum to make 50 vibrations in a minute?

$$\frac{140850}{2500} = 56.34 \text{ inches.}$$

Given the length of a pendulum, to find how many vibrations it will make in a given time.

Bring the given time into seconds; then, as the given length of the pendulum is to $39\frac{1}{8}$, so is the square of the given time to the square of the number of vibrations, the square root of which is the number sought.

Example.—If the length of a pendulum be 48 inches, how many vibrations will it make in a minute?

The given time is 60 seconds.

$$48 : 39.125 :: 60^2 : \frac{39.125 \times 3600}{48} = \frac{140850}{48} = 293{,}438,$$

the square root of which is 54.17 vibrations in a minute.

CENTRE OF OSCILLATION AND PERCUSSION.

92. The distance of the centre of oscillation or percussion of any compound pendulum from its centre of suspension, is equal to the sum of the products of each

more bodies, or of one body from the figure and extent of which we are not permitted to abstract.

89. The centre of oscillation of a compound pendulum is a point in it at such a distance from the centre of suspension, that a simple pendulum, of a length equal to that distance, will have the same angular velocity with the compound pendulum itself.

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Bring the given time of the number of vibr given number of se required pendulum

body into the square of its distance from the centre of suspension, divided by the sum of the products of each body into its distance from that centre.

Thus, if any number of bodies, A, B, C, &c. and their respective distances from the centre of suspension, a, b, c, &c. be given, then the distance of the centre of oscillation from the centre of suspension is $\frac{Aa^2+Bb^2+Cc^2}{Aa+Bb+Cc}$.

Let $A = 4$ lbs. and its distance from the centre of suspension 4 inches, $B = 6$ lbs. and its distance 2 inches, and $C = 8$ lbs. and its distance from the same point 3 inches; then, $\frac{4\times 4^2+6\times 2^2+8\times 3^2}{4\times 4+6\times 2+8\times 3} = \frac{64+24+72}{16+12+24} = 3\frac{1}{13}$; that is, the centre of oscillation is $3\frac{1}{13}$ inches from the centre of suspension.

The distance of the centres of oscillation and percussion from the axis of motion is as follows, where the axis of motion is at the vertex and in the plane of the figure:—

In a right line, small parallelogram, and cylinder, $\frac{2}{3}$ the axis of the figure.

In a triangle, $\frac{3}{4}$ the axis.

In the parabola, $\frac{5}{7}$ of the axis.

If a cylinder, of which the altitude is a, and the radius r, be suspended from its vertex, the distance of the centre of oscillation from the vertex is $\frac{2a}{3} + \frac{r}{2a}$.

If a cone be suspended from the vertex, the altitude of which is a, and the radius of the base r, the distance of the centre of oscillation from the vertex is $\frac{4a}{5} + \frac{r^2}{5a}$.

In a sphere, r = radius, d = distance of the axis of motion from its centre; then the distance of the centre of oscillation from the axis of motion is $d + \frac{2r^2}{5d}$.

If the sphere be suspended by a point in its surface, then the distance of the centre of oscillation from that point is $\frac{7r}{5}$.

Ex. 1.—If a cylinder, of which the radius is 4 inches, and altitude 1 foot, be suspended by its vertex; required the length of a simple pendulum which will vibrate in the same time.

$$\text{Here, } r = 4 \text{ inches, } a = 12.$$

$$\therefore \frac{2a}{3} + \frac{r}{2a} = \frac{2 \times 12}{3} + \frac{4}{2 \times 12} = 8\tfrac{1}{6} \text{ inches.}$$

Ex. 2.—If a globe, the radius of which is 6 inches, be suspended by a point in its surface, required the length of a simple pendulum which will vibrate in the same time.

$$\text{Here } r = 6 \text{ inches, and } \frac{7r}{5} = \frac{7 \times 6}{5} = 8\tfrac{2}{5} \text{ inches.}$$

If the globe be suspended by a string 8 inches long, attached to a point in its surface, then the length of the pendulum is $d + \frac{2r^2}{5d}$.

Since $r = 6$, we have $d = 8 + 6 = 14$;

$$\therefore d + \frac{2r^2}{5d} = 14 + \frac{2 \times 6^2}{5 \times 14} = 15\tfrac{1}{35} \text{ inches.}$$

ON VIS VIVA AND WORK.

The *vis viva* of a body is its mass multiplied by the square of its velocity.

Work,* or dynamical effect, as it is sometimes called,

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* Poisson observes that the principle of virtual velocities gives immediately the conditions of the equilibrium of force, applied to machines, whilst that of *vis viva*, or living forces, includes equally the theory of their motion, and furnishes the most direct means of calculating the effects of the forces

...ppose a body moved, and a resistance overcome [illegible] them, without the other, is insufficient to [illegible] ...rk. The work, produced by a pressure moving a body through a certain space, is defined to be the product [illegible]

...ich are applied to them. "Machines," says he, "may be defined as instruments, or systems of solid bodies, adapted to transmit the action of forces from one part to another of the system."

To calculate the effects of machines in motion, Poisson gives the following equation:—

$$\tfrac{1}{2}\Sigma m v^2 - \tfrac{1}{2}\Sigma m k^2 = \int \Sigma P\, dp - \int \Sigma Q\, dq.$$

Where v is the velocity of the mass m, at the end of any time t; k its original velocity, or that corresponding to $t = 0$; P, any one of the pressures applied to produce motion in the machine; and Q, any one of the resistances opposing its motion; dp, the elementary space passed over by P, in the direction of its force, in the time dt; and dq, the space passed over in the same time by Q, in the direction of its force. The integrals to be taken so as to vanish at the commencement of the motion.

Poisson also observes, that $P\,dp$, and $Q\,dq$, have received different denominations; Coriolis proposes to call them quantities of elementary work. Adopting this, as Poisson has done, the integrals $\int \Sigma P\,dp$, and $\int \Sigma Q\,dq$, will express the whole motive work, and the whole resisting work, from the commencement of motion up to the time under consideration.

The above equation shows that the increment, during any time, of half the sum of the living forces of all the parts, is equal to the excess of motive work over the resisting work in the same time.

If the machine had no original velocity, and we wish to find the friction as a separate retarding force, then

$$\tfrac{1}{2}\Sigma m v^2 = \int \Sigma P\, dp - \int \Sigma Q\, dq - \int \Sigma f N\, ds;$$

where f is the co-efficient of friction; N, the mutual pressure of the rubbing surfaces; ds, the elementary space described by their point of contact; and $\int \Sigma Q\, dq$, the useful work done by the machine.

Professor Moseley's formula, which agrees with that of Poisson, is

$$\Sigma U_1 = \Sigma U_2 + \Sigma u + \frac{1}{2g}\Sigma w (v_2^2 - v_1^2).$$

Here ΣU_1 is the number of units of work done by the moving power; ΣU_2 is the units of useful work done; Σu, the work expended in overcoming friction, and other resistances, opposed to the motion of the machine; v_1 and v_2 the velocities at the commencement and termination of the time of working; and $\frac{1}{2g}\Sigma w (v_2^2 - v_1^2)$ is half the aggregate difference of the vires vivæ of the various moving parts of the machine, at

from multiplying the pressure by the space through which this pressure acts.

UNIT OF WORK.—The unit of work, in this country, in terms of which we measure any amount of work, is the work done where a pressure of one pound is exerted through one foot, the pressure acting in the direction in which the space is described; if, instead of one pound being moved through one foot, it be moved through two feet, it is clear that the work is doubled, or that two units of work have been done; and if through m feet, m units. Now if, instead of one pound pressure exerted through m feet, two pounds were exerted through m feet, the work would be doubled, or $2m$ units of work would be done; and if n times, then n times m units of work would be done. Hence, generally, if a pressure of n pounds be exerted through m feet, then the number of units of work will be represented by n times m, supposing the pressure to be exerted in the same direction as the space described.

The units of work being represented by U, we have

$$U = mn \quad . \quad . \quad . \quad (1)$$

$$n = \frac{U}{m} \quad . \quad . \quad . \quad (2)$$

$$m = \frac{U}{n} \quad . \quad . \quad . \quad (3)$$

From these equations, any two being given, the others may be determined.

The ablest and most extensive writers on the subject of work, are Poncelet and Professor Moseley. In the

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the commencement and termination of the above-named time. See "Mechanical Principles of Engineering and Architecture," page 161.

In all practical cases we have to calculate the work of machines only when they have a uniform, or periodic motion; in either case, $\frac{1}{2g} \Sigma w (v_2^2 - v_1^2) = 0$, and the above equation becomes

$$\Sigma U_1 = \Sigma U_2 + \Sigma u.$$

From this equation we have

$$V_{\prime}^2 - V^2 = \pm \frac{2gU}{W} \quad . \quad . \quad . \quad (5)$$

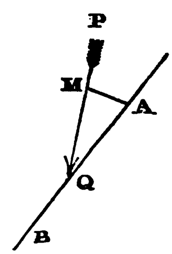
This evidently applies to the case where a body is projected either downwards or upwards with a given velocity the work done, in the former case, is

$$U = \tfrac{1}{2}\frac{W}{g}(V_{\prime}^2 - V^2) = \tfrac{1}{2}(V_{\prime}^2 - V^2)M \quad . \quad . \quad . \quad (6)$$

and, in the latter,

$$U = \tfrac{1}{2}(V^2 - V_{\prime}^2)M \quad . \quad . \quad . \quad (7)$$

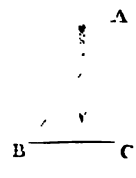
93. *To estimate the work in the case where the pressure is not exerted in the direction in which the space is described.*



Suppose the point of application Q of the pressure to traverse the line AB, the pressure being constantly exerted, not in the direction from A towards B, but in a direction which is always parallel to the line MQ; then will the work done while Q is moving from A to Q be the same as though the point of application, instead of moving from A to Q, had moved along the line MQ, from M to Q; QM being the projection of AQ upon the direction of the pressure; so that the work done would be represented by $P \times MQ$; P being the pressure in lbs., and MQ the space in feet. For, resolve P into two forces, one perpendicular, and the other parallel to AB; as no motion takes places in the perpendicular direction, no work is done in that direction, therefore all the work is done by the resolved part of P, in the direction AB; call this R, then the work done $= R \times$

AQ; if $PQA = \theta$, we have $R = P \cos. \theta$; $\therefore R \times AQ = P.AQ \cos. \theta$; but $AQ \cos. \theta = MQ$, hence the work done $= P.MQ$, AM being perpendicular to PQ.

The theory of the inclined plane presents an application of this principle. Let AB represent the surface of an inclined plane, upon which is placed a weight W, which is to be raised up that inclined plane. And let it be required to determine the units of work in raising it. In this case, the direction of the weight W to be overcome is obviously not that in which the motion takes place: the body moves from B to A; the direction of the weight is always parallel to AC, whence it follows, by the preceding proposition, that the work which must be done to raise it is the same as would be necessary to raise it through the projection of the space which it describes in the direction AC, parallel to which the pressure W is exerted: then the work done in raising it from B to A is represented by $W \times AC$. Let us now suppose that this inclined plane is subject to friction: then the units of work necessary to overcome the friction must be added to the number of units represented by $W \times AC$. For the sake of simplicity, we may suppose the inclination to be small, as in most practical cases it is, then the resistance opposed by friction is obviously represented by the weight $W \times$ the coefficient of friction—call this coefficient f: then friction $= W \times f$. Now, the direction of this friction is parallel to the surface of the plane, acting from A towards B, as the body is drawn up. The work expended upon the friction is then represented by the friction in lbs. $\times$ by the distance in feet, or, $W \times f \times AB$. Call L the length, and H the height; then, $W \times f \times L =$ work expended on friction, and $W \times H =$ work due to gravity: $\therefore W \times f \times L + W \times H$ represents the whole units of work expended in rai the body up the incline, or $W (f \times L + H)$.



If the body had been descending instead of ascen

...ad been required to ascertain the work which must ... to let it down the incline, then we have

$$W \times H - WLf = W\{H - Lf\} \quad . \quad . \quad (8)$$

...t U = units of work in each case, then, generally,

$$U = W \times H \pm WLf = W\{H \pm Lf\}$$

$$= W \times L\left\{\frac{H}{L} \pm f\right\} \quad . \quad . \quad . \quad (9)$$

This may be done in a more scientific manner as follows:—

The forces which act on the body are the resolved part of the weight, and the friction acting up the plane.

The resolved force in the direction of the plane is g sin. a, and the force perpendicular to the plane is g cos. a; and, since the friction is proportional to the pressure, we have g cos. a, multiplied by the coefficient of friction equal to the resistance from friction, or fg cos. a = friction, this being the force retarding the body in its descent, and g sin. a = force urging the body down the plane; hence, the whole force urging the body down the plane is g sin. $a - fg$ cos. a. Let F represent this force, then, by formula for constant forces, we have

$$v^2 = 2Fs;$$

$$\therefore v^2 = 2(g \sin. a - fg \cos. a)s;$$

$$\frac{v^2}{2} = (g \sin. a - fg \cos. a)s.$$

Multiply both sides of this equation by the mass M, and we have

$$\frac{Mv^2}{2} = Mg . s \sin. a - Mfgs \cos. a.$$

The left-hand side of this equation is half the *vis viva*, and, since the work done is equal half the *vis viva*,

[illegible]

$\therefore$ the work [illegible]

$= W$ [illegible]

If [illegible] be the whole length [illegible] B the base, then

[illegible]

substitute these values [illegible]

Work $=$ [illegible]

but [illegible]

$\therefore$ work $=$ [illegible]

This shows [illegible] the weight of the [illegible] the inclined plane.

The work [illegible] as that [illegible] base of the inclined [illegible] an inclined plane [illegible] site to the motion [illegible] therefore the [illegible] sented by

[illegible]

Hence, generally

[illegible]

This may be [illegible]

[illegible]

This result differs [illegible] we [illegible] small as it usually [illegible] (13 [illegible] plane.

If the body be de[illegible] hat the [illegible]

[illegible] and if the inclination be such th[illegible] [illegible] 0, and no work would in this case b[illegible] [illegible] the body down, or it would in that ca[illegible] [illegible] of its own accord.

[illegible] of the plane is such that th[illegible] [illegible] (9) becomes

$$[illegible] \quad H \quad . \quad . \quad . \quad (14)$$

[illegible] the power moving the bod[illegible] [illegible] a body whose weig[illegible] [illegible] whose height is H, an[illegible] [illegible]

[illegible] body in pounds, by th[illegible] [illegible] and again by the sufficient [illegible] [illegible] weight in pounds by th[illegible] [illegible] Subtract the latter produ[illegible] [illegible] remainder will be the units [illegible] [illegible] down the plane.

[illegible] to move a bod[illegible] [illegible] instead of th[illegible] [illegible]

[illegible] with a give[illegible] [illegible] before th[illegible] [illegible]

[illegible] accumulated in th[illegible]

work required to stop the train must also be equal to half the *vis viva*.

Let M = mass, $V_{,}$ = velocity, $P_{,}$ = the resistance, and S = the space; then

$$\tfrac{1}{2} MV_{,}^2 = PS \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore S = \tfrac{1}{2}\frac{MV_{,}^2}{P} \quad . \quad . \quad . \quad . \quad (2)$$

Neglecting the resistance of the atmosphere, the train will be stopped by the resistance from friction.

The resistance from friction is $f.\,W = f.Mg$; substitute this for P in the above equation, and we have

$$S = \frac{MV_{,}^2}{2fMg} = \frac{V_{,}^2}{2fg} \quad . \quad . \quad . \quad (3)$$

To find the time:

When a body moving with a velocity $V_{,}$, and acted on by a retarding force, so that in the time t it has a velocity V, we have equation (2), page 122,

$$V = V_{,} - Ft = V_{,} - \frac{P}{M}\, t;$$

$$\text{since } F = \frac{P}{M}.$$

But when the train stops, $V = 0$;

$$\therefore V_{,} = \frac{P}{M}.t;$$

$$\therefore t = \frac{M}{P}.V_{,} \quad . \quad . \quad . \quad (4)$$

But, as before, friction $= fMg$,

$$\therefore t = \frac{M}{fMg}.V_{,} = \frac{V_{,}}{fg} \quad . \quad . \quad (5)$$

Examples on Forces and Work.

1. There is an inclined plane of 1 in 90, and a locomotive is required to ascend it at the rate of 20 miles per hour, with a tender, the whole weighing 40 tons; the

friction on the incline = 6 lb. per ton. At what horse-power must the engine work to ascend the inclined plane at the required rate?

$$20 \text{ miles per hour} = \tfrac{1}{3} \text{ of a mile per minute};$$
$$= 1760 \text{ feet per minute.}$$

The rise being 1 foot in 90 feet, the rise in 1760 feet

$$= \frac{1760}{90} = 19.55 \text{ feet.}$$

The units of work without friction

$$= 19.55 \times 40 \times 2240 = 1751680.$$

The units of work expended on friction

$$= 6 \times 40 \times 1760 = 422400;$$

whole units of work $= 1751680 + 422400 = 2174080$;

$$\text{and } \frac{2174080}{33000} = 65.88 \text{ horse power.}$$

2. At what rate will a locomotive ascend an inclined plane of 1 in 110 with a load of 100 tons: the horse-power of engine being = 60, the friction upon the incline = 6 lbs. per ton?

Let x = the velocity per minute in feet; then height that the load is raised to in x feet

$$= \frac{x}{110};$$

$\therefore$ units of work without friction $= \dfrac{x}{110} \times 100 \times 2240$

$$= 2036.36\,x.$$

And units of work expended on friction

$$= x \times 6 \times 110 = 660\,x;$$

Therefore whole number of units of work expended

$$= 2696.36\,x;$$

But the number of units of work done by the engine

$$= 60 \times 33000 = 1980000;$$

$$\therefore\ 2696.36\,x = 1980000;$$

$$\therefore\ x = \frac{1980000}{2696.36}$$

$$= 735 \text{ feet per minute.}$$

Therefore rate per hour $= 735 \times 60 = 44100$ feet;

$$= 8.35 \text{ miles per hour.}$$

3. How many units of work must a fire-engine do to throw A cubic feet of water per minute to a height of H feet?

The weight of the water $= A \times 62.5$;

$$\therefore \text{ work} = A \times 62.5 \times H.$$

$$\text{The horse-power} = \frac{A.H.62.5}{33000}.$$

4. Let A represent the area of the section of a stream in square feet, and V, the mean velocity of the water in feet per second. Required the number of units of work which the stream is capable of doing per minute, at a given point of its course.

The volume of water which passes through the section A in 1 minute, and which does the work, is represented by $60\,A.V.$ The weight W of this water is therefore $60\,A.V.62.5$. Now, the mass having this weight has a velocity V, it has therefore an amount of accumulated work represented by

$$\tfrac{1}{2}\,\frac{60\,A.V.62.5}{32.17}.V^2, \text{ or } \frac{60 \times 62.5}{64\frac{1}{3}}.A.V^3;$$

$$\therefore \text{ the work} = 58.28\,A.V^3.$$

5. There is a stream moving with a velocity of V feet per second, and whose section is A square feet at the point where its velocity is V; a dam is erected across the stream, and a fall obtained of H feet. What work may be obtained from this fall of water per minute, and what its equivalent in horse-power?

Since $V^2 = 2gH$

[illegible] $= 62.5 \times 60 \ldots A \cdot V \cdot H.$

[illegible] $= 62.5 \times 60 \ldots V$

$$[illegible] = \frac{62.5 \times 60 \ldots V \cdot H}{33000}.$$

[illegible] stream with section A square feet, and [illegible] at the point where its section is measured [illegible]. There is a fall of H feet from this [illegible] it is required by means of an over-shot wheel [illegible] from the stream h feet above the higher level. [illegible] water can be raised per second, supposing that [illegible] over-shot wheel is $\frac{1}{n}$th part?

In this example the velocity of the water in feet [illegible] is $\frac{5280}{60^2} N$; $\therefore \frac{5280 \times 62.5 \, A N}{60^2} =$ lbs. of [illegible] through A per second $=$ lbs. of water [illegible] which therefore produce a power of [illegible] units per second. Now, let x be the [illegible] feet raised per second through $(H + h)$

[illegible] to raise it $= 62.5 \, x \, (H + h)$;

$$[illegible] \frac{5280}{60^2} \cdot \frac{A \cdot H \cdot N}{n} = 62.5 \, x \, (H + h);$$

$$[illegible] \frac{5280}{60^2} \cdot \frac{A \cdot H \cdot N}{n \, (H + h)}.$$

[illegible] the water is raised from the lower [illegible] the case in which the water is [illegible]

[illegible] cubic feet raised through h feet

$\therefore$ [illegible]

which [illegible]

$\therefore$ [illegible]

the [illegible]

$\therefore$ the [illegible] $=$ [illegible]

Now, the work [illegible] $=$ [illegible]

$\therefore$ [illegible] $=$ [illegible]

[illegible]

7. A body whose [illegible] zonta[illegible] plane [illegible] force of [illegible] how many feet will [illegible]

By formula [illegible] $=$ [illegible] $F = \frac{P}{W}$ [illegible] deduct[illegible]

$P =$ [illegible]

$f =$ [illegible]

$\therefore S =$ [illegible]

8. There is a [illegible] weighing [illegible] zontal plane [illegible] how many feet will [illegible] per [illegible]?

Since $V^2 = 2gH$;

$$\therefore \tfrac{1}{2}\frac{W}{g}V^2 = \tfrac{1}{2}\frac{W}{g}2gH = WH = 62.5 \times$$

Since $W = 62.5 \times 60 A . V$;

$$\therefore \text{horse-power} = \frac{62.5 \times 60 A . V}{33000}.$$

6. There is a stream with section A sq with mean velocity at the point where its sec by N miles per hour. There is a fall of H stream, and it is required by means of an to raise water from the stream h feet above How much water can be raised per second. the work of an over-shot wheel is $\frac{1}{n}$ th part

In this example the velocity of the per second is $\frac{5280}{60^2}N$; $\therefore \frac{5280 \times 62.5}{60^2}$ water which pass through A per second which fall, and which therefore pr $\frac{62.5 \times 5280}{60^2}AHN$ units per second. number of cubic feet raised per second feet;

$\therefore$ the work necessary to raise it =

$$\therefore \frac{62.5 \times 5280}{60^2} \cdot \frac{A.H.N}{n} = 62.5$$

$$\therefore x = \frac{5280}{60^2} \cdot \frac{A.H.N}{n(H +}$$

This case is when the water is raise level. Let us now take the case in raised from the upper level.

Let x be the number of cubic feet per second;

And the work expended on friction

$$= 6\tfrac{1}{2} \times 80 \times 300 = 156000;$$

$$\therefore \text{ whole effective work } = 441333\tfrac{1}{3};$$

the work necessary to raise the empty train without friction $= 14 \times 2240 \times 3\tfrac{1}{3} = 104533\tfrac{1}{3}$;

$$\text{work expended on friction } = 6\tfrac{1}{2} \times 14 \times 300 = 27300;$$

$$\therefore \text{ whole work expended on empty train } = 131833\tfrac{1}{3}.$$

Therefore the work accumulated in the two trains

$$= 441333\tfrac{1}{3} - 131833\tfrac{1}{3} = 309500;$$

$$\therefore 309500 = \tfrac{1}{2}\frac{W}{g} . V^2;$$

$$\text{but } W = 80 + 14 = 94 \text{ tons} = 210560 \text{ lbs.};$$

$$\therefore \tfrac{1}{2} . \frac{210530}{32\frac{1}{6}} . V^2 = 309500,$$

$$\text{or } \frac{631680}{193} . V^2 = 309500;$$

$$\therefore V^2 = 94.56;$$

$$V = \sqrt{94.56} = 9.7 \text{ feet per second.}$$

Let x = distance the heavier train will run along the horizontal plane; then $x \times 80 \times 8$ = work expended on friction, which in this case is equal the work accumulated in the train.

$$\therefore x \times 80 \times 8 = \tfrac{1}{2}\frac{W}{g} . V^2;$$

$$640\,x = \tfrac{1}{2} \times \frac{80 \times 2240}{\frac{193}{6}} \times 94.56 = \frac{537600 \times 94.56}{193}$$

$$= 411.5 \text{ feet.}$$

Or thus:—

The friction being 8 lbs. per ton, the coefficient f will be $\frac{8}{2240} = \frac{1}{280}$; then, by equation (3), page 139,

$$\therefore\ S = \frac{V_{\prime}^{2}}{2fg} = \frac{94.56}{2 \times 32\frac{1}{6} \times \frac{1}{280}} \quad \frac{94.56 \times 280 \times 3}{193} = 411.5,$$

the same as before.

$$\text{By equation (5) } t = \frac{V_{\prime}}{fg} = \frac{9.7}{\frac{1}{280} \times \frac{193}{6}}$$

$$= \frac{9.7 \times 280 \times 6}{193} = 83.4 \text{ seconds.}$$

Let x = distance the lighter train will ascend up the incline, then the vertical height the train ascends $= \frac{x}{90}$;

$\frac{x}{90}(14 \times 2240) + x \times 14 \times 6\frac{1}{2}$ = the units of work;

but the work done is equal half the *vis viva*.

$$\therefore\ \frac{x}{90}(14 \times 2240) + x \times 14 \times 6\tfrac{1}{2} = \tfrac{1}{2}\,\frac{W}{g}\,.\,V^{2}$$

$$= \frac{1}{2} \times \frac{14 \times 2240}{32\frac{1}{6}} \times 94.56;$$

$$\frac{31360}{90}x + 91\,x = \frac{1}{2} \times \frac{31360 \times 94.56}{\frac{193}{6}};$$

$$\therefore\ 439.44\,x = \frac{8896204.8}{193};$$

$$x = 104.9 \text{ feet.}$$

In drawing materials from any given depth, we may take into account the weight of the rope, by proceeding in the following manner:—

14. From what depth can a weight of 6 cwt. be raised by a rope 4 inches in circumference, and with 3 horse-power, in 6 minutes?

$$\text{Formula } U = \frac{\mu l^2}{2} + Wl;$$

$$3 \times 33000 \times 6 = \frac{.736 l^2}{2} + 672 l;$$

$$.368 l^2 + 672 l = 594000;$$

$$l^2 + 1826 l = 1614130.434;$$

$$l^2 + 1826 l + 833569 = 2447699.434;$$

$$l + 913 = 1564.5; \therefore l = 651.5 \text{ ft.} = 108.5833 \text{ fathoms.}$$

15. A winding engine is observed to raise to the surface a weight of material = 12 cwt. in $6\frac{1}{2}$ minutes, from a depth of 115 fathoms; the rope which raises it is a flat one formed of 3 cylindrical ropes, each 3 inches in circumference: what is the working horse-power of the engine?

$$\text{Formula } U = \frac{\mu l^2}{2} + Wl;$$

$$\mu = .046 \times 9 \times 3 = 1.242;$$

$$6\tfrac{1}{2}\, U = \frac{1.242 \times 476100}{2} + 1344 \times 690;$$

$$= \frac{591316.2}{2} + 927360;$$

$$= 1223018.1;$$

$$U = \frac{1223018.1}{6.5} = 188156 \text{ units of work};$$

$$\frac{188156}{33000} = 5.7 \text{ horse-power.}$$

16. Suppose the engine in the preceding example has a cylinder $20\frac{1}{4}$ inches in diameter, and makes a stroke of 2 feet 10 inches, and 15 strokes per minute, what is the pressure per square inch on the piston?

$$\begin{array}{r} 20.5 \\ 20.5 \\ \hline 420.25 \\ .7854 \\ \hline 330.06435 \\ 42.5 \\ \hline 14027.734675 \\ \hline \end{array} \quad 42.5 = (2\text{ ft. }10\text{ in.}) \times 15$$

Units of work = 188156 from preceding example;

$$\therefore \frac{1888156}{14027.73} = 13.4 \text{ pressure per square inch on piston.}$$

17. A train of 90 tons ascends an incline of 1 in 500: find the uniform speed when the horse-power of the engine is 60.

Let x = rate in feet per hour, then, as the rail rises 1 foot in 500, its rise in x feet will be $\frac{x}{500}$.

The work done by gravity per hour is

$$90 \times 2240 \times \frac{x}{500};$$

and the work done by friction $= 90 \times 8 \times x$;

$\therefore$ the whole work required to be done by the engine

$$= 90 \times 2240 \times \frac{x}{500} + 90 \times 8 \times x;$$

work of engine per hour $= 60 \times 33000 \times 60$.

Now, when the motion is uniform the work done by the engine must be equal to the work of the resistance;

$$\therefore 90 \times 2240 \times \frac{x}{500} + 90 \times 8 \times x = 60 \times 33000 \times 60;$$

$$x = 105769 \text{ feet, or } 20 \text{ miles per hour.}$$

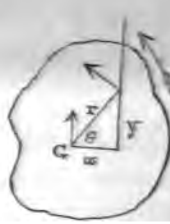
To find the Vis Viva of a body under any circumstances.

95. There are three cases:—When the body has only a motion of translation, or when all the particles of the body describe parallel lines;—when the body revolves round a fixed centre;—and when there is a combination of these two kinds of motion, viz. both angular motion, and motion round a fixed centre.

When the body has a motion of translation, every particle of the body moves with the same velocity; the *vis viva* is therefore equal to the mass of each particle multiplied by the square of the velocity; ∴ it equals the mass of the whole body multiplied by the $(\text{velocity})^2$.

To find the Vis Viva of a body which turns round a fixed centre.

Let there be a body turning round a centre of motion G.



Let m be one of the particles.

r = its distance from G.

a = the angular velocity with which the body turns round G.

The angular velocity of the body is the same as the angular velocity of the radius, and the linear velocity of the particle m is ra,* and therefore the *vis viva* $= m\,r^2a^2$. So also with any other particle $m_{,}$ at a distance $r_{,}$ from C, the *vis viva* $= m_{,}\,r_{,}^2\,a^2$. So with any particle m_2 at a distance r_2, &c. Adding the *vis viva* of the particles together, we get the *vis viva* of the whole body

$$= (m r^2 + m_{,} r_{,}^2 + m_2 r_2^2 + \&c.)\, a^2;$$

but $mr^2 + m_{,}r_{,}^2 + m_2r_2^2 + \&c.$ = the moment of inertia, and the *vis viva* of a rigid body turning round the centre

* Since the angular velocity of a body is the velocity at an unit of distance from the centre of motion, the linear velocity is the angular velocity multiplied by the whole distance.

of motion equals the moment of inertia multiplied by the square of the angular velocity.

The work done = $\frac{1}{2}$ (the moment of inertia) × the square of the angular velocity.

Professor Moseley puts $m_{,}$ for the volume of a particle, and μ for the weight of a unit of a volume; then

$$m_{,}\mu = \text{weight};$$

$$\text{and mass} = \frac{m_{,}\mu}{g};$$

$$\text{also for a particle } m_2 \text{ the mass} = \frac{m_2\mu}{g} \text{ \&c.};$$

these substituted for m_1, m_2, &c. we get the

$$\textit{vis viva} = a^2\left(\frac{\mu}{g}\right)I;$$

$$\text{hence the work} = \tfrac{1}{2} \text{ the } \textit{vis viva} = \tfrac{1}{2}a^2\left(\frac{\mu}{g}\right).I;$$

$$\text{if } U = \text{work, } U = \tfrac{1}{2}a^2\left(\frac{\mu}{g}\right).I;$$

$$a^2 = \frac{2g}{\mu}\cdot\frac{U}{I}.$$

If the body, instead of setting out from rest, had been moving with an angular velocity $a_{,}$, then the work accumulated between the first and second positions of the body is

$$\tfrac{1}{2}a^2\left(\frac{\mu}{g}\right).I - \tfrac{1}{2}a_{,}^2\left(\frac{\mu}{g}\right).I;$$

$$\therefore\ a = a_{,} \pm 2\left(\frac{g}{\mu}\right).\frac{U}{I}.$$

The sign ± to be taken according as the motion is accelerated or retarded.

To find the Vis Viva when there is a combination of the motions of translation and rotation.

Let figure (page 152) represent any body; G, its centre of gravity; m, one of the particles of which the body is composed.

suppose that G is moving along, and at the same body turns round the centre of gravity. Let G ng in the direction of the arrow with a velocity v, body at the same time is turning round with an velocity a.

t r represent the distance of m from the centre of gra y.

Draw x perpendicular to the direction of v.

Draw y parallel the direction of v.

And let $\theta = \angle$ that r makes with x.

Then, if the body had no motion of rotation, the particle m would move with the velocity v, in the vertical direction y. But the body has a motion of rotation in addition to the motion of translation; if the centre of gravity were fixed, the particle m would have a velocity ra, in the direction perpendicular to r; and therefore, the body having two velocities simultaneously impressed on it, will, by the second law of motion, produce both these velocities.

To find the resultant of these velocities, we must resolve them horizontally and vertically.

Now, ra resolved vertically $= ra \cos.\theta$; and ra resolved horizontally $= ra \sin.\theta$.

Since the angle of the direction which ra makes with y is evidently $= \theta$, we must add to the vertical part the velocity v;

$\therefore$ whole vertical velocity of the particle $= ra \cos.\theta + v$; and the whole horizontal velocity of the particle $= ra \sin.\theta$.

The resultant of these velocities

$$= r^2a^2 \cos.^2\theta + 2vra\cos.\theta + v^2 + r^2a^2\sin.^2\theta \text{ (Art. 10)};$$

$$= r^2a^2 + v^2 + 2vra\cos.\theta;\ \text{since } \sin.^2\theta + \cos.^2\theta = 1;$$

$$= r^2a^2 + 2vax + v^2;\ \text{since } v\cos.\theta = x;$$

$$\therefore \text{the } vis\ viva = m(v^2a^2 + 2vax + v^2).$$

So also for any particle $m_{,}$, the *vis viva* $= m_{,}(r_{,}^2a^2 + 2vax_{,} + v^2)$; and for any other particle m_2 the *vis viva*

$= m_2(r_2^2\omega^2 + 2v\omega x_2 + v^2)$; and so on for all the particles.

Adding these together we get the whole vis viva.

Now, the sum of the first column

$$= \omega^2 (mr^2 + m_1 r_1^2 + m_2 r_2^2 + \&c.)$$
$$= \omega^2 \text{ (moment of inertia)}.$$

The sum of the second column

$$= 2v\omega (mx + m_1 x_1 + m_2 x_2 + \&c.):$$

but, by a property of a centre of gravity (see Art. 12),

$$mx + m_1 x_1 + m_2 x_2 + \&c. = 0:$$

and this is the reason why the centre of gravity is here taken as the fixed point round which the motion of rotation is supposed to take place.

The sum of the third column

$$= v^2 (m + m_1 + m_2 + \&c.);$$
$$= v^2 \text{ (whole mass of the body)};$$

$\therefore$ the whole *vis viva* = (the moment of inertia) ω^2 + (mass) v^2.

$\therefore$ when a body has a motion of rotation and a motion of translation, the *vis viva* = (moment of inertia) $\times$ (angular velocity)^2 + (mass) (velocity centre of gravity)^2; but when a body has a motion of translation, the *vis viva* = (mass) (velocity)^2;

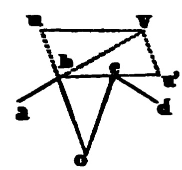
And when a body has a motion of rotation, the *vis viva* = (mass) (angular velocity)^2;

$\therefore$ when a body has a motion of rotation and a motion of translation, the *vis viva* equals the *vis viva* due to the translation + the *vis viva* due to the rotation.

CENTRIFUGAL FORCE.

To find the Centrifugal Force of a body when its dimensions are small compared with its distance from the centre, the velocity being constant.

96. Let a body whose weight is W be attached to a rigid rod AC, and suppose a velocity be communicated to it in the direction AB, perpendicular to AC; if the body was free it would, by Art. 88, continue its motion in the direction AB, but, being connected by a rigid rod, it must necessarily describe a circle round C as a centre. The two forces, one of which tends to draw the body to the centre, and the other tends to move it from that centre, are called the *centripetal* and *centrifugal* forces respectively.



Now, as we suppose the body to be small compared with its distance from the centre, we may consider it as a material point moving with a velocity V, and we may substitute for the circle a regular polygon of an infinite number of equal sides, then the body describes each side of the polygon with the same velocity, that is, the primitive velocity V is the same when the body passes from one side of the polygon to the other. For, since the centrifugal force acts always perpendicularly to the motion of the body, it does no work, and can destroy neither the living force nor the velocity of the body. Now, ub or vu' is the very small velocity generated by the centrifugal force F, during the small time t which the body requires to move over the elementary side of the polygon; then, if M be the mass of the body, we have (since the force is measured by the mass, multiplied by the velocity and divided by the time)

$$F = \frac{M \times vu}{t};$$

Join bo, then the triangles obc, vbu, and bvu' are isosceles and similar, for the angle abo = the vertical angle vbu; also the exterior angle obc = interior and opposite angle buv. Now, since ob bisects the angle abc, the angle abo = the angle obc; hence the angle obc = the angle vbu. But the angle obc = the angle buv; hence the angle vbu = the angle buv, or the triangle vbu is isosceles and similar to the triangle obc. The triangle vbu is equal in all respects to the triangle bvu'; therefore the triangles obc, vbu, and bvu' are similar;

$$\therefore \; bo : bc :: bu' : vu';$$

$$vu' = \frac{bc \times bu'}{bo} = \frac{bc \times V}{R} \quad . \quad . \quad . \quad (1)$$

where R represents the radius of the circle, and bu' the primitive velocity V; besides, the side bc of the polygon described by the body in the indefinitely small time t;

$$\therefore \; bc = Vt;$$

this substituted in the above, gives

$$vu' = \frac{V^2 t}{R};$$

$$\therefore \; F = \frac{M \times vu'}{t} = \frac{M \times V^2 \times t}{R \times t} = \frac{M \cdot V^2}{R};$$

$$\text{or, since } M = \frac{W}{g},$$

$$F = \frac{W}{g} \cdot \frac{V^2}{R} \cdot \quad . \quad . \quad . \quad (2)$$

Centrifugal Force of a body of finite dimensions.

97. Let $ABCD$ represent a thin slice of the mass revolving round O, with an angular velocity $a_{,}$, the angular velocity of an elementary mass m of this slice will be $ra_{,}$; r being

circumference of axle $= 4 \times 3.14159 = 12.56636$ inches;

$\therefore$ if $n =$ number of revolutions before stopping,

distance described by bearing point of axle

$$= (12.56636 \,.\, n) \text{ inches};$$

$$= n\,(1.04386) \text{ feet};$$

and weight of wheel $= 450 \left\{ (3.14159 \times 9) \times 1 \times \frac{3}{4} \right.$

$$\left. + \, 8\left(3.14159 \times 9 \times \frac{10}{3}\right) + \left(3.14159 \times \frac{6}{9} \times \frac{3}{8}\right) \right\}$$

$$= 1413.7155 \times \left\{ \frac{27}{4} + 240 + \frac{1}{3} \right\} = 1413.7155 \times 252.083$$

$$= 356374.1156 \text{ lbs.};$$

$\therefore$ work expended on friction $= n\,(1.04386)$

$$\times\ (356374.1156) \times .05;$$

$$= n\,(18600.234);$$

and this must $=$ work done by weight $+$ work done by rope;

$$= 201600 + \frac{1}{2} \times .042\, c^2 h^2;$$

where $c =$ circumference in inches;

and $h =$ length of rope in feet.

$$\therefore n = \frac{201600 + .021 \times 25 \times 8100}{18600.234} = \frac{205852.5}{18600.234}$$

$$= 11.067 \text{ turns.}$$

2d. $f\,(1.04386) \times (356374.1156) \times 5 = 205852.5$;

$$\therefore f = \frac{41170.5}{372004.684} = .1106 = \text{co-efficient of friction.}$$

A rectangular mass of cast iron, 10 feet long and 1 foot square in section, is movable round a horizontal axis 3 feet from one end: what angular velocity will it have acquired in descending from an elevation of 45° to a horizontal position?

The moment of inertia of this rectangular mass about an axis passing through its centre of gravity, and parallel to one of its edges $= \frac{1}{12} abc\,(b^2 + c^2) = I.$

Now, the moment of inertia of body about an axis parallel to this, and distant from it (h) feet $= h^2 m + I = I_{\prime}$;

$$m = abc;$$

$$\therefore\ I_{\prime} = (12h^2 + b^2 + c^2)\frac{abc}{12};$$

$$\left.\begin{array}{rl} \text{In this instance } a & = 1 \\ b & = 1 \\ c & = 10 \\ h & = 2 \end{array}\right\}$$

$$I_{\prime} = \left(12(2)^2 + 1 + 100\right)\frac{10}{12};$$

$$= \left(\frac{1490}{12}\right) = 124.16.$$

The work accumulated in the body in its descent

$$= abc\,.\,\mu\,.\,s;$$

where s = space passed through by the centre of gravity

$$= 2 \sin. 45°;$$

$$\therefore \text{work} = 10 \times 450 \times \sqrt{2};$$

$$= 4500 \times 1.414 = 14.14 \times 450.$$

$$\text{But this work} = \frac{1}{2}\,\textit{vis viva} = \frac{1}{2}\,a^2\,.\,\frac{\mu}{g}\,.\,I_{\prime};$$

$$\therefore\ 14.14 = \frac{1}{2}\,a^2\,\frac{1}{\frac{193}{6}} \times 124.16 = \frac{372.48}{193}\,a^2;$$

$$\therefore\ 1364.51 = 186.24\,a^2;$$

When a cone revolves about an axis parallel to its axis of symmetry, and at a distance from it = radius of the base; what is the centrifugal force, supposing the cone of cast-iron 3 feet high, $1\frac{1}{2}$ feet in diameter, and made to revolve 120 times per minute?

$$F = \frac{1}{3}\frac{a^2}{g}\pi a^3 b\mu;$$

(here $a = 4\pi$ = the $\angle$ passed through per second);

$$= \frac{1}{3}\frac{(16\pi^2)^3}{g}a^3 b\mu = \frac{1}{3} \times \frac{16}{\frac{193}{6}} \times (3.1416)^3 \times \left(\frac{3}{4}\right)^3 \times 3 \times 450;$$

$$= \frac{96}{193} \times 31.0065 \times \frac{27}{64} \times 450 = \frac{31.0065}{12.8666} \times 1215;$$

$$= 2.4098 \times 1215 = 2927.907 \text{ lbs.}$$

If the same cone revolves at the same velocity round an axis coinciding with its side; find the centrifugal force.

$$\text{In this case } F = \frac{1}{4}\frac{a^2}{g}\frac{\pi a^3 b^2 \mu}{\sqrt{a^2 + b^2}};$$

$$= (\text{the expression in last case}) \times \frac{3}{4}\frac{b}{\sqrt{a^2 + b^2}};$$

$$= 2927.907 \times \frac{3}{4} \cdot \frac{3}{\frac{3}{4}\sqrt{5}};$$

$$= 2927.907 \times \frac{3}{\sqrt{5}} = 2927.907 \times \frac{3}{2.236};$$

$$= 3928.494 \text{ lbs.}$$

In the two preceding examples suppose the cone to be of cast-iron 1 foot high, and base = 1 foot in diameter, and made to revolve 30 times per minute: find centrifugal force.

(1.) $$F = \frac{1}{3} \cdot \frac{a^2}{g} \cdot \pi a^3 b\mu.$$

Here $a = \frac{1}{2}$ foot; $b = 1$ foot; $w = 450$ lbs.;

and v = space passed through in a second = π:

$$F = \ldots \times \frac{\pi^2}{193} \times \left(\frac{1}{2}\right) \ldots \times 450 = \frac{2}{193}$$

$$\times \frac{31.0065}{8} \times 450;$$

$$= \frac{13952.925}{772} = 18.074 \text{ lbs.}$$

$$= \text{the former expression} \times \frac{3}{4} \cdot \frac{b}{\sqrt{a^2 + b^2}};$$

$$= 18.074 \times \frac{3}{4} \cdot \frac{1}{\ldots} = 18.074 \times \frac{3}{2} \cdot \frac{1}{2.236};$$

$$= 18.074 \times \ldots$$

... diameter, and ... a vertical ... what is its

$$V = \frac{100}{60} 2\pi R = \frac{5}{3} \times 2 \times 3.1416 \times \frac{3}{2};$$
$$= 5 \times 3.1416;$$
$$= 15.708;$$
$$R = \frac{3}{2};\ g = \frac{193}{6};$$
$$\therefore\ F = \frac{265.0725 \times (15.708)^2}{\frac{3}{2} \times \frac{193}{6}};$$
$$= \frac{65394.33}{\frac{193}{4}} = 1355.3.$$

A shaft of cast-iron, whose section is 8 inches by 4 inches, and whose length is 4 feet, revolves round an axis 5 inches in diameter, and placed at a distance of 3 inches from one extremity; the two surfaces of contact are wrought-iron upon cast-iron, unctuous: find the horse-power expended upon the friction of this axis by reason of the centrifugal force, when it revolves 200 times in a minute?

$$F = \frac{WV^2}{gR};$$
$$W = \text{weight of body} = \frac{8}{12} \times \frac{4}{12} \times 4 \times 450$$
$$= \frac{2}{3} \times \frac{1}{3} \times 4 \times 450;$$
$$= 2 \times 4 \times 50 = 8 \times 50 = 400 \text{ lbs.}$$
$$V = \text{velocity per second} = \frac{200}{60} \times 2\pi R;$$
$$= \frac{400}{60} \pi R = \frac{20}{3} \pi R;$$
$$R = \text{radius of circle described by centre of gravity};$$
$$= 2 \text{ ft.} - 3 \text{ in.} = 24 - 3 = \frac{21}{12} = \frac{7}{4} \text{ feet};$$

$$W = 9 \times 112 \times 20 = 9 \times 2240 = 20160;$$

$$V = 30 \times \frac{5280}{60 \times 60} = \frac{.5280}{120} = 44 \text{ feet per second};$$

$$R = 400 \times 3 = 1200;$$

$$\therefore F = \frac{20160 \times (44)^2}{\frac{193}{6} \times 1200} = \frac{20160 \times 1936}{193 \times 200}$$

$$= \frac{1951488}{1930} = 1011.13.$$

Now, if A at the extremity of the outer rail be the point from which the moments are measured, and $F = 5$ feet from A, where the force is collected, and G the middle of the rails, the carriage will be overthrown or not according as $AF \times F >$ or $< AG \times W$.

$$AF = 5 \text{ ft.}; \ F = 1011.13 \text{ lbs.};$$

$$AG = 2 + \frac{4.5}{12} = 2.375; \ W = 20160 \text{ lbs.}$$

$\therefore$ it will be overthrown or not according as

$$5 \times 1011.13 > \text{ or } < 20160 \times 2.375;$$

or as $5055.65 >$ or < 47880;

$\therefore$ it will not be overthrown.

A piece of iron is placed without any support on the interior circumference of the rim of a wheel, at a distance of 6 feet from the axis, about which the wheel revolves horizontally: what number of revolutions must it make per minute, in order to prevent the iron from falling off?

That there may be an equilibrium, the friction produced by the centrifugal force must be equal the weight of the body; so that if f be the co-efficient of friction, $fF = W$;

$$\text{but } F = \frac{W \cdot V^2}{gR} = \frac{W \cdot (aR)^2}{gR} = \frac{W a^2 R}{g};$$

$$\therefore \frac{f W a^2 R}{g} = W;$$

$$\therefore a^2 = \frac{g}{f R};$$

$$\therefore a = \sqrt{\frac{g}{f R}};$$

If n be the number of revolutions per minute, $2 \pi n$ will be the space described per minute at distance unity from the axis, so that $\frac{2 \pi n}{60}$, or $\frac{\pi n}{30}$, will be the space described per second at the same distance, or the angular velocity a;

$$\therefore a = \frac{\pi n}{30} = \sqrt{\frac{g}{f R}};$$

$$\therefore n = \frac{30}{\pi} \sqrt{\frac{g}{f R}}.$$

$$f = 16, \; R = 6;$$

$$\therefore n = \frac{30}{3.1416} \sqrt{\frac{\frac{193}{6}}{6 \times 16}};$$

$$= \frac{30}{3.1416} \sqrt{\frac{193}{36 \times 16}};$$

$$= \frac{30}{3.1416} \sqrt{\frac{193}{576}}.$$

$$= \frac{30}{3.1416} \cdot \sqrt{.335}$$

$$= \frac{30}{3.1416} \times .578$$

$$= \frac{17.34}{3.1416}$$

$$= 5.52$$

Two balls, one 6 inches and the other 10 inches in diameter, are fixed to the extremity of a rod 3 feet in length; this rod is fixed horizontally on a vertical spindle, at a distance of 27 inches from the centre of the smaller ball (or at a distance of 2 feet from circumference of the same ball). Find the centrifugal force of these balls of cast-iron, when they are made to revolve 100 times per minute.

Let W_1 = weight of smaller ball;

W_2 = weight of larger ball.

$$\text{Now } W_1 = \frac{4}{3}\pi a_1^3 \mu;$$

$$W_2 = \frac{4}{3}\pi a_2^3 \mu;$$

$$F = \frac{W_2}{g} \frac{V_2^2}{R_2} - \frac{W_1}{g} \cdot \frac{V_1^2}{R_1};$$

Now let m = number of revolutions per second;

$$\text{then } V_1 = 2\pi m R_1;$$

$$V_2 = 2\pi m R_2;$$

$$\therefore V_1^2 = 4\pi^2 m^2 R_1^2;$$

$$V_2^2 = 4\pi^2 m^2 R_2^2.$$

$$\text{Now } m = \frac{100}{60} = \frac{5}{3};$$

$$R_1 = 27 \text{ inches} = \frac{9}{4} \text{ foot};$$

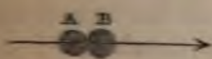
$$R_2 = 17 \text{ inches} = \frac{17}{12} \text{ foot};$$

$$a_1 = 3 \text{ inches} = \frac{1}{4} \text{ foot};$$

$$a_2 = 5 \text{ inches} = \frac{5}{12} \text{ foot};$$

IMPACT.

98. To describe the nature of the action that takes place in the impact of two bodies.



Let A, B, be two bodies, which we shall suppose to be moving in the same straight line, in the direction represented by the arrow, A being supposed to overtake B, and to strike upon it. Then, when A comes in contact with B, a mutual pressure will be exerted between the two bodies, tending to retard the motion of A, and to accelerate that of B. This mutual pressure will act equally upon each body; therefore action and reaction are equal and opposite. The effect of this mutual pressure will be to diminish the velocity of A, and to increase that of B, until the two bodies are made to move with equal velocities. As soon as the velocities become equal the bodies will cease to press upon each other, and, if no further action took place between the bodies, they would continue to move on with the same velocity.

But a further action will take place in the following manner: — Since the bodies have been pressing upon each other, and since all bodies have some degree of softness, it is clear that they will be in a state of compression at the time the velocities become equal: but all compressed bodies have a tendency, greater or less, to recover their shape; therefore the two bodies will begin to recover their shape immediately after the time the two velocities become equal, and on this account they will begin to press on each other again. The effect of this renewed pressure will be to increase the velocity of B, and diminish that of A; and this will continue till the velocity of B becomes so much greater than that of A that the two bodies separate, and then all further action ceases.

In the first place, we may observe that this action consists of two distinct parts; one, that during which the bodies are undergoing compression, and the other that in which the bodies are expanding so as to recover their shape. The former is called the *action of compression*, and the latter the *action of expansion*.

We must also observe that the action of compression ceases, and that of expansion begins, at the instant when the velocities of the two bodies become equal.

The instant at which the bodies first come in contact is called the *beginning of impact*.

The instant at which the velocities become equal is called the *end of compression*; or, what is the same thing, the *beginning of expansion*.

And the instant at which the bodies separate is called the *end of impact*.

If the bodies strike each other moving in opposite directions the description would be the same.

On the difference between Finite and Impulsive Forces.

99. A *finite* force is a force that acts for a *finite* time, such as the force of gravity. An *impulsive* force is a force which acts during an extremely small time, and is so called because the forces that take place in any impulse, or impact, always act in an extremely small time.

In the case of the impact just described, the time which elapses between the beginning and end of impact is always extremely small, so small that it neither has been nor can be measured.

In consequence of this, all that we can find out in the se of impact is the whole effect produced by the impact. We cannot find how the body is moved during the pact, because the time is so short that it is impossible to reciate the variation of motion.

he contrary is the case with finite forces. When we

are concerned with finite forces, it is always our object to find out how the body is moving at any time while it is under the action of a finite force. This constitutes the great distinction between problems which relate to finite and impulsive forces.

There is also another important difference between them, which is this: in the case of finite forces, the body moves over a finite space during the action of the force, and therefore its position is sensibly changed by the force; but, in the case of impulsive forces, the action of the forces is finished before the position of the bodies has been sensibly altered.

100. To prove that no momentum is lost in the impact of the bodies A and B (the momentum of a body is its mass multiplied by its velocity).

The pressure exerted by A upon B is equal to the pressure exerted by B upon A, during the whole impact.

Now, by the *third* law of motion, a pressure always generates or destroys a momentum proportional to itself; therefore the momentum which is destroyed in A by the pressure of B must be equal to the momentum generated in B by the pressure of A;

$\therefore$ during the whole of the impact whatever momentum A loses, at any instant, B gains; consequently, no momentum is lost;

i. e. the sum of the momenta of two bodies is invariable.

101. Having given the velocities of A and B at the commencement of the impact, to find their common velocities at the end of compression.

Let u and v be the velocities of A and B at the beginning of the impact, and u' v' their velocities at the end of compression, and

$$\therefore u' = v'.$$

Let A and B denote the masses of A and B, then the

... two bodies ... the beginning

... two bodies at the end

... B ...

... is always the same ...

... moving in the sa...

... apply to the c...

... direction.

... opposite direct...

... impact, then

... impact will b...

which produced their compression. It is also found by experiment that the pressure exerted during the expansion of the two impinging bodies, A and B, is always in a given proportion to the pressure exerted during their compression; and, consequently, it follows that the momentum produced in either of the bodies during the expansion is in a given proportion to the momentum produced in that body during the compression.

To express this mathematically:

Let u'' and v'' be the velocities of A and B, at the end of the impact:

Then the momentum generated in B, during the compression, is $B v' - B v$; (because v' is the velocity at the end, and v at the beginning of compression.)

In the same way, the momentum generated during the expansion $= B v'' - B v'$; and these two must be in a given proportion;

$\therefore \dfrac{B v'' - B v'}{B v' - B v}$ = a given quantity, which quantity is generally represented by the Greek letter λ;

$$\therefore \frac{B v'' - B v'}{B v' - B v} = \lambda;$$

$$\therefore v'' - v' = \lambda (v' - v);$$

which equation represents what is called the *law of elasticity*.

In the same way precisely it may be shown that

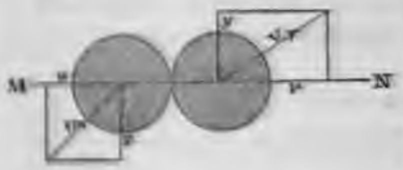
$$u'' - u' = \lambda (u' - u).$$

The quantity λ, which is used here, is called the *co-efficient of elasticity*, and is always less than unity; though in some cases, as glass, it is very nearly $= 1$.

When $\lambda = 1$ the elasticity is said to be perfect; and when $\lambda = 0$ the body is said to be perfectly inelastic.

OBLIQUE IMPACT.

103. To investigate the motion of two bodies after oblique impact.



Let $A B$ be the two bodies, and suppose that before impact they move with velocities U, V, which are represented by the diagonals of the parallelograms in the figure. Let $M N$ be the direction in which the bodies strike each other, *i.e.* the line which is perpendicular to the surfaces of both bodies at the point where they come in contact. The bodies are supposed to be perfectly smooth, and therefore they can only press upon each other in the direction of the line $M N$.

Resolve the velocity U into two other velocities u and x, u in the direction $M N$, and x in a direction perpendicular to it.

In the same way resolve the velocity V into v and y; v in the direction $M N$, and y in the direction perpendicular to it.

Then the velocities x and y will not be altered by the impact; because the pressure exercised by the impact is perpendicular to the direction of x and y, and pressure can never produce any effect in a direction perpendicular to itself.

Also, by the second law of motion, the alteration produced in the velocities u and v will be exactly the same as if the velocities x and y did not exist.

Hence the results obtained in the case of direct impact will apply to find the alteration produced in the velocities u and v. If therefore u' and v' be the velocities of the bodies at the end of compression in the direction $M N$ (and $\therefore u' = v'$), and if u'' v'' be the velocities at the end of the

impact in the direction MN, then by the previous proposition (101),

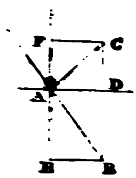
$$u' = v' = \frac{Au + Bv}{A + B};$$

$$\text{and } u'' - u' = \lambda(u' - u);$$

$$v'' - v' = \lambda(v' - v).$$

IMPACT ON A PLANE.

104. Suppose the body to strike the plane AD.



Let the line AB represent the velocity with which it strikes the plane. Resolve the velocity AB into two other velocities, AD and AE; AD along the plane, and AE perpendicular to it. Then the velocity AD will not be altered by the impact (therefore it takes place perpendicular to the plane); but the velocity AE will be altered, in exactly the same way as if the velocity AD did not exist. At the moment of compression the body will have no velocity in a direction perpendicular to the plane; (because at the end of compression the two striking bodies have the same velocity, and, since the plane has no velocity, the body must have no velocity either.) Let AF be the velocity generated in the expansion, then AF must $= \lambda\,(AE)$ (102.)

Hence we may find the whole motion after impact, by simply remembering that the velocity of the body along the plane is not altered, and the velocity of the body perpendicular to the plane is reversed in direction, and diminished in the proportion of $\lambda : 1$.

$$\text{Let } \angle\, BAE = \alpha;$$

$$\angle\, CAF = \beta;$$

$$\text{then } \tan.\,\alpha = \frac{EB}{EA};$$

$$\tan. \beta = \frac{CF}{FA};$$

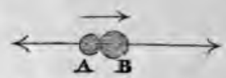
ce $EB = CF$, and $FA = \lambda\,(AE)$,

$$\therefore \tan. \alpha = \lambda \tan. \beta.$$

α is called the angle of incidence; β the angle of reflection;

$\therefore$ the tangent of the angle of incidence $= \lambda$. (the tangent of the angle of reflection.)

105. The force of gravity is measured by the momentum produced in the body in one second of time; if, however, *impulsive* force were measured that way it would be infinitely great, and therefore we measure impulsive force by the whole momentum it produces during the time of its action.



106. Let A, B be two bodies moving in the direction of the upper arrow; and let A overtake B. During the time of compression there is a *mutual impulse* acting between the bodies, and we shall measure it as an impulsive force.

Let R be the impulsive pressure that acts between these bodies.

R then is measured by the momentum it produces during the time of its action, *i.e.* during the compression.

Let u = velocity of A before impact;

u' = velocity of A at the end of compression;

$\therefore u - u'$ = alteration of velocity produced by compression;

$\therefore A\,(u - u')$ = alteration of momentum.

This is produced by the force R;

$\therefore$ this must $= R$;

$$\therefore A\,(u - u') = R \;.\;.\;.\;.\;.\; (1)$$

In the same way the alteration of momentum in B

$$= B(v' - v);$$

$$\therefore B(v' - v) = R \ldots\ldots (2)$$

$$\therefore \text{from } (1)$$

$$B(v' - v) = A(u - u');$$

$$\therefore u' = \frac{Au + Bv}{A + B} \text{ as before (101.)}$$

In the case of the expansion, we know from the law of elasticity (102) that the effect produced by the expansion : that produced by the compression :: $\lambda : 1$.

$\therefore$ if R be the impulsive force during compression, λR = impulsive force during expansion;

and the alteration of momentum during the expansion

$$= A(u' - u'');$$

$$\therefore A(u' - u'') = \lambda R;$$

and in the same way $B(v'' - v') = \lambda R$;

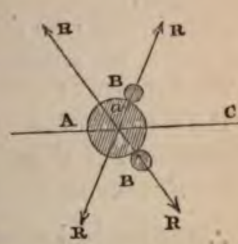
$$\therefore A(u' - u'') = \lambda A(u - u');$$

$$\therefore u' - u'' = \lambda(u - u');$$

$$\therefore u'' = (1 + \lambda)u' - \lambda u$$

and also $v'' = (1 + \lambda)v' - \lambda v$

107. Let a ball A strike two other equal balls B and B, which are at rest; the directions of the impulses making equal angles with the line $A\,C$.



Let R be the impulsive force between the ball A and the balls B during compression.

Let θ be the angle which the direction of the impulse makes with $A\,C$; let u be the velocity of A before impact; u' the velocity of A at the end of compression; v' that of each of the balls B at the same time.

...ident that ... takes place along the line ...

... during the compression ...

... are equivalent to a force ...

... of momentum produced during ...

... velocity before impact, and ...

... velocity ...

... in it during that time

... during the compres...

substituting the value of R from (2) in (1);

and from (3) $v' = u' \cos.\theta$;

we have $A(u - u') = 2Bu' \cos.^2\theta$;

$$\therefore u' = \frac{Au}{A + 2B\cos.^2\theta};$$

$$v' = u' \cos.\theta;$$

$$= \frac{Au\cos.\theta}{A + 2B\cos.^2\theta};$$

The velocity at the end of expansion may be found from the law of elasticity.

Let u'' = velocity of A at the end of expansion;

by (102) $u' - u'' = \lambda(u - u'')$;

and, in the same way, if v'' be the velocity of the balls B at the end of expansion,

$$v'' - v' = \lambda v';$$

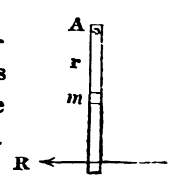
(because B has no original velocity, *i. e.* $v = 0$);

which gives $u'' = (1 + \lambda)u' - \lambda u$;

$$v'' = (1 + \lambda)v';$$

substituting the value above found for u' and v' in these equations, we obtain the velocities at the end of expansion.

To find the effect of the Impact when a body having a fixed axis is struck by an impulsive force.



108. Let the accompanying figure represent a beam, having a fixed point or axis at A, and let it be struck by an impulsive force R, acting at the end of the beam in a direction perpendicular to it.

be any particle of the beam at a distance r from point A, and let a be the angular velocity pro- the impulsive action.

the velocity of the particle m produced by the will be $r\,a$, and the momentum produced in it refore be mra.

momentum therefore represents the impulsive force called into action on the particle m, and similarly we may find the impulsive force brought into action on any other part of the beam.

Now, we want to find the tendency of the forces to turn the beam round A, and therefore we must consider the moment round that point.

The moment of the force on the particle m round A will be got by multiplying the force by r.

$$\therefore \text{ the moment of the impulsive force} = m\,r^2\,a;$$

$$\text{so also for any other particle } m' \;.\;.\;.\; = m'r_{,}^2a;$$

and so on.

Therefore, adding these together, we find the whole moment of the forces which tend to turn the beam round the point A, which moment therefore is

$$= (m\,r^2a + m_{,}r_{,}a\;.\;{}^2.\;.\;.\;+ \&c.\;.\;.\;.\;.)$$

$$= a(m\,r^2 + m_{,}r_{,}^2\;.\;.\;\;\;.\;+ \&c.\;.\;.\;.\;.)$$

The quantity here in brackets is the moment of inertia of the beam round the point A (86);

$\therefore$ if MK^2 be that moment of inertia, the moment of all the forces tending to turn the beam round its axis is

$$MK^2\,a.$$

Now, the force R is that force which tends to turn the beam about the point, and produces those forces whose moment we have just estimated; therefore the moment of the force R about the point A must be the same as that above found.

But the moment of $R = Rb$;

(b being the length of the beam;)

$$\therefore MK^2 a = Rb;$$

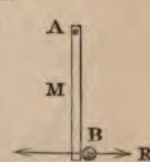
$$\therefore a = \frac{Rb}{MK^2};$$

$\therefore$ the angular velocity *produced* by the impulsive force is the moment of the impulsive force divided by the moment of inertia.

We have supposed that the beam has no angular velocity previously to the action of the force; we will now suppose that it had originally a certain angular velocity, and that the effect of the force is to alter that velocity into $a_{,}$; then $a_{,} - a$ is the alteration of velocity produced by the impact, or, in other words, is the angular velocity produced by the force, and

$$\therefore a_{,} - a = \frac{Rb}{MK^2}.$$

Example.—The Ballistic Pendulum.—The ballistic pendulum is an instrument for measuring the velocity of a cannon ball, *i. e.* the force of gunpowder. It consists, in its simplest form, of a beam, which can swing on a fixed axis at one end, while the ball strikes the other end; and the angle through which that end moves being known, the velocity of the cannon ball may be computed.



Let the figure represent a beam moveable round a fixed axis at A, and let a ball B strike it at its lowest point; (the beam is supposed perfectly hard.)

Let R be the impulsive force that is exercised by the ball and the beam.

Let $a_{,}$ be the angular velocity produced in the beam;
v the velocity of the ball before impact;
v' the velocity of the ball after impact.

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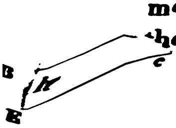


But the work done by gravity = (the weight of the beam) × (the height the centre of gravity rises)

$$\text{Let } BE = h;$$

and since the centre of gravity rises a space $= \frac{h}{2}$;

$$\therefore \frac{1}{2} M K^2 a_{,}^2 = M g \frac{h}{2};$$

$$(\because M g \text{ is weight of beam});$$

$$\therefore M K^2 a_{,}^2 = M g h \quad . \; . \; (5)$$

To find h we must know CE or c.

The extent of the vibration is known by fixing a sharp needle into the end of the beam, which gives a scratch to any soft composition laid in a groove; from this we have the chord of the arc described, and by cor. prop. 8 of the sixth book of Euclid,

$$b h = c^2;$$

$$\therefore h = \frac{c^2}{b}.$$

Substitute this in equation (5), and we find $a_{,}$; substitute again the value of $a_{,}$ thus found in equation (4), and we find v, the velocity of the cannon ball.

by the previous proposition

$$MK^2 a_{,} = Rb \dots\dots (1)$$

(b being the length of the beam, and M its mass ;)

and $v - v'$ is the velocity generated, and $B(v - v')$ is the momentum produced in the ball by the force R;

$$\therefore B(v - v') = R \dots\dots (2)$$

and also at the end of compression (which here is the end of impact, because the beam is *hard*), the ball and the point of the beam which it touches have the same velocity.

The velocity of the end of the beam is $b a_{,}$;

$$\therefore b a_{,} = v' \dots\dots (3)$$

Now, from (3) we have v';

$$\therefore \text{substituting in (2) } B(v - b a_{,}) = R;$$

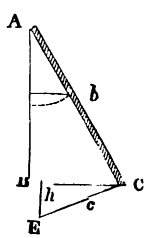
$$\therefore \text{from (1) } \{MK^2 + Bb^2\} a_{,} = Bbv \dots\dots (4.)$$

If we suppose v to be given, we can find from this the angular velocity of the beam.

If the angular velocity of the beam is known, the velocity of the ball may be found.

This latter is what is required.

The angular velocity of the beam may be easily found as follows:—



If the height to which the beam rises be observed, the angular velocity with which it starts may be known.

Work accumulated to raise it to that height $= \frac{1}{2}$ the *vis viva.*

Now, the *vis viva* $= MK^2 a_{,}^2$ (95.)

When the beam has risen to the point C, gravity overcomes the force which raised it to that height, and therefore the work done by gravity equals the work accumulated in the beam.

But the work done by gravity = (the weight of the beam) × (the height the centre of gravity rises)

$$\text{Let } BE = h;$$

and since the centre of gravity rises a space $= \dfrac{h}{2}$;

$$\therefore \frac{1}{2} MK^2 a_{,}^2 = Mg\frac{h}{2};$$

$$(\because Mg \text{ is weight of beam});$$

$$\therefore MK^2 a_{,}^2 = Mgh \quad . \; . \quad (5)$$

To find h we must know CE or c.

The extent of the vibration is known by fixing a sharp needle into the end of the beam, which gives a scratch to any soft composition laid in a groove; from this we have the chord of the arc described, and by cor. prop. 8 of the sixth book of Euclid,

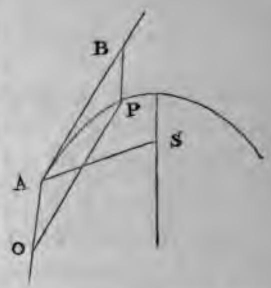
$$bh = c^2;$$

$$\therefore h = \frac{c^2}{b}.$$

Substitute this in equation (5), and we find $a_{,}$; substitute again the value of $a_{,}$ thus found in equation (4), and we find v, the velocity of the cannon ball.

ON PROJECTILES.

109. *To find the path described by a body which is projected with a given velocity, and in a given direction, neglecting the resistance of the air.*



Suppose the body to be projected in the direction AB, with the velocity v, and let t be the time in which the body would describe AB with the velocity of projection; and let AO be the space through which the force of gravity would cause it to descend in the same time t; complete the parallelogram $ABPO$, and P will be the point in the curve where the body will be found after the time t.*

Now, since the velocity in the direction AB is uniform,

$$AB = tv \dots\dots\dots (1)$$

and, by page 98,

$$AO = BP = \tfrac{1}{2}gt^2 \dots (2)$$

$$\text{from (1) } t^2 = \frac{AB^2}{v^2};$$

$$\therefore BP = \tfrac{1}{2}g \cdot \frac{AB^2}{v^2};$$

$$\text{or } AB^2 = \frac{2v^2}{g} \cdot BP;$$

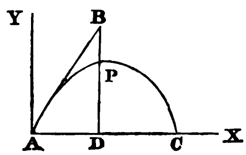
---

* After the proof of the parallelogram of forces, a formal proof of the parallelogram of velocities seems unnecessary; for, since the lines which represent the proportion of the forces are described in the same time, and since the velocity of a body is proportional to the space described in a given time, these lines will also represent the velocities. Thus as the velocity with which the diagonal is described is to the velocity with which either of the sides is described, so is the diagonal to either of the sides.

$$\text{or } PO^2 = \frac{2v^2}{g} . AO \ldots (3)$$

hence the curve is a parabola whose parameter is $\frac{2v^2}{g}$.

110. *To find the path described by a Projectile referred to horizontal and vertical axes of co-ordinates.*



Let $AD = x$, $DP = y$, $BAC = \phi$; then, since $AB = tv$, we have

$$AD = x = tv \cos. \phi,$$

$$\text{and } BD = tv \sin. \phi;$$

but BP, by the last article, $= \frac{1}{2} gt^2$,

and $PD = y = BD - BP = tv \sin. \phi - \frac{1}{2} gt^2$;

$$i.e. \ x = tv \cos. \phi \ldots\ldots\ldots (4)$$

$$y = tv \sin. \phi - \tfrac{1}{2} gt^2 \ldots (5);$$

from (4) $t = \frac{x}{v \cos. \phi}$; substitute this in (5), and we have

$$y = x \tan. \phi - \tfrac{1}{2} . \frac{gx^2}{v^2 \cos.^2 \phi} \ldots . (6.)$$

If h be the height due to the velocity v, then $h = \frac{v^2}{2g}$ by page 98; this, substituted in (6),

$$y = x \tan. \phi - \frac{x^2}{4h \cos.^2 \phi} \ldots\ldots (7.)$$

111. *To find the range and time of flight of the Projectile, on a horizontal plane.*

The range AC can be easily determined by equation (6), for, when $x = AC$, y is evidently $= 0$; hence, making $y = 0$ in (6), we have

$$0 = x \tan.\phi - \tfrac{1}{2} \cdot \frac{gx^2}{v^2 \cos.^2\phi};$$

$$\therefore \frac{gx^2}{2v^2 \cos.^2\phi} = x \tan.\phi;$$

$$\text{or } x = \frac{2v^2 \cos.^2\phi \cdot \tan.\phi}{g} = \frac{2v^2 \sin.\phi \cos.\phi}{g};$$

$$\text{or, since } 2 \sin.\phi \cos.\phi = \sin. 2\phi,$$

$$x = AC = \frac{v^2 \sin. 2\phi}{g} = 2h \sin. 2\phi \quad . \; . \; . \quad (8.)$$

The horizontal range AC will evidently be greatest when $\sin. 2\phi$ is greatest, that is, when $\sin. 2\phi = 1$, or $2\phi = 90$; $\therefore \phi = 45$ will give the greatest range for a given velocity.

At all angles equally distant from $45°$ the range is the same; for, since

$$\sin.(90° + w) = \sin.(90° - w);$$

$$\text{or } \sin. 2\left(45° + \frac{w}{2}\right) = \sin. 2\left(45° - \frac{w}{2}\right);$$

we may either put $45° - \frac{r}{2}$, or $45 - \frac{w}{2}$, for ϕ, and the range will be the same.

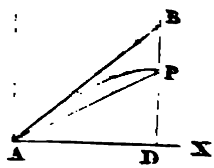
The time of flight can be determined by equation (5), in making $y = 0$:

$$0 = tv \sin.\phi - \tfrac{1}{2} gt^2;$$

$$t = \frac{2v \sin.\phi}{g};$$

112. *The velocity and direction of a projectile being given, to find the time of flight and range on an inclined plane, passing through the point of projection.*

Let AP be the inclined plane, a = angle PAD, and ϕ, as before, the angle of projection.



Sin. BAP : sin. BPA :: BP : AB;

but sin. APB = sin. APD = cos. PAD
= cos. a:

and sin. BAP = sin. $(\phi - a)$, and BP
$= \frac{1}{2} gt^2$, and $AG = tv$;

$$\therefore \sin.(\phi - a) : \cos. a :: \tfrac{1}{2} gt^2 : tv :: \tfrac{1}{2} gt : v;$$

$$\therefore t = \frac{2v}{g} \cdot \frac{\sin.(\phi - a)}{\cos. a}.$$

To find the range.

$$\text{Sin. } ABP : \sin. APB :: AP : AB.$$

$$\text{Now, } AB = tv = \frac{2v^2}{g} \cdot \frac{\sin.(\phi - a)}{\cos. a} = 4h \cdot \frac{\sin.(\phi - a)}{\cos. a};$$

and, as before, $APB = \cos. a$;

also, sin. $ABP = \cos. BAD = \cos. \phi$;

$$\therefore \cos. \phi : \cos. a :: AP : \frac{4h \cdot \sin.(\phi - a)}{\cos. a}:$$

$$\text{hence the range } AP = \frac{4h \cdot \sin.(\phi - a) \cdot \cos. \phi}{\cos.^2 a}.$$

If AP falls below AD, then a must be taken negative.

Examples.

1. From the two ends of a vertical line, two bodies are, at the same instant, projected towards each other, with velocities v, v'. Required their distance when half the time in which they would meet is elapsed.

2. With what velocity must a body be projected from a tower, in a direction parallel to the horizon, so that it shall strike the ground at a distance from the foot of the tower, equal to half the height of the tower?

3. A body, projected at an angle of 15° to the horizon, ranges 100 feet on an horizontal plane. How high would

it rise, if it were projected straight upwards with the velocity of projection?

4. A body is projected up an inclined plane, whose length is 10 times its height, with a velocity of 30 feet in 1″. In what time will its velocity be destroyed?

5. A body falls 20 feet down an inclined plane whose height is one-sixth of its length. What is the time of its motion, and what its last acquired velocity?

6. A body is projected at an angle of 45° with a velocity of [illegible] feet per second. Find its horizontal range.

7. A stone projected at an angle of 30° strikes the horizontal plane at the distance of 50 yards. With what velocity was it projected?

8. A body is projected at an angle of 60° to the horizon, with a velocity of 100 feet per second. What will be its time of flight before it strikes a plane inclined to the horizon at an angle of 30°? and what will be its greatest [illegible] above the plane?

9. [illegible] and B, are projected perpendicularly [illegible] and b). It is required to assign [illegible] common centre of gravity [illegible]

[illegible] direction of projection: find [illegible] above the inclined [illegible] the greatest height [illegible]

[illegible] perfect elasticity as [illegible] above a per- [illegible] rebound continually till [illegible] the whole space [illegible]

[illegible] same time with [illegible] of a vertical [illegible] in the middle [illegible] acquired in [illegible]

[illegible] same time from dif-

ferent points in the same vertical line, and with the velocities acquired move along a horizontal plane, till one overtakes the other. Show that the time of *A's* descent is equal to the time of *B's* uniform motion.

14. Two bodies are projected towards each other in the same vertical plane from two given points, so as to describe the same parabola. Find the point where they meet.

15. A body projected at an angle of 60° hits a mark at the distance of 300 feet upon an inclined plane whose elevation is 30°. Required the velocity of projection, the greatest height above the plane, and the time of flight.

16. An imperfectly elastic ball being projected from P, a point in the periphery of a circle PQR, whose centre is C, after impinging at Q and R, returns to P. Required the value of the angle CPQ.

17. In the oblique impact of an imperfectly elastic body upon a plane, cotan. incidence : cotan. reflection :: force of compression : force of elasticity.

18. A projectile is to be thrown across a plain 120 feet wide, to strike a mark 30 feet high, the velocity of projection being that acquired down 80 feet. Find at what angle it must be projected.

19. A ball is projected from a given point in the horizontal plane at an angle of 30°, and after describing two-thirds of its horizontal range, strikes against a sonorous body; having given the whole interval between the instant of projection, and the instant when the sound reaches the point of projection: to find the initial velocity.

20. A ball, whose elasticity : perfect elasticity :: n : 1, is projected obliquely upwards, from a point in the horizontal plane, upon which it impinges and rebounds continually. Prove that the ranges and times of flight in the successive parabolas described form geometric progressions; and find their sum.

HYDROSTATICS.

[illegible] Hydrostatics treats of the pressure, weight, and equilibrium of non-elastic fluids.

If a fluid be at rest in a vessel, the base of which is parallel to the horizon, equal parts of the base are equally pressed by the fluid.

For, upon every part of the base there is an equal column of the fluid supported; and, as all the columns are of equal weight, they must press the base equally, or equal parts of the base will sustain an equal pressure.

All the parts of the fluid press equally at the same depth.

The pressure of a fluid at any depth is as the depth of the fluid.

For the pressure is as the weight, and the weight is as the height of a column of the fluid.

[illegible] by its own weight, or by any other [illegible] presses equally in every direction [illegible]

[illegible] nature of fluidity, which is, to yield [illegible] direction. If it cannot give way to [illegible] applied, it will press against other parts [illegible] direction of that force; and the pressure [illegible] same: for if any one was less, the [illegible] until the pressure was the same [illegible]

[illegible] a fluid, the pressure is [illegible] against the sides, or even [illegible]

[illegible] base of the containing [illegible] altitude, whatever [illegible]

The pressure of a fluid against any upright surface, as the gate of a sluice, is equal to the area of that surface multiplied by half its depth.

Ex.—If the gate of a sluice be 9 feet broad and 6 feet deep, what is the pressure of water against it?

$9 \times 6 \times 3 = 162$ = the area multiplied by half the depth.

$$\frac{162 \times 1000}{16} = 10125 \text{ lbs. or } 4.5 \text{ tons.}$$

The pressure against the internal surface of a cubical vessel is three times the weight of the fluid contained in it; for the pressure against one side of the vessel is equal to half the pressure on the bottom.

Ex.—What pressure does the internal surface of a cubical vessel sustain, each side of the cube being 4 feet?

$$4^3 \times 62\tfrac{1}{2}^* = 4000 = \text{pressure on the bottom.}$$

$$4 \times 4 \times \frac{4}{2} \times 62\tfrac{1}{2} = 2000 = \text{pressure on one of its sides;}$$

$$\text{hence } 4 \times 2000 = 8000 = \text{pressures on the four sides;}$$

$$\therefore \text{ the whole pressure} = 8000 + 4000 = 12000.$$

If a hollow sphere be filled with fluid, the whole pressure against the internal surface is three times the weight of the contained fluid.

Let r = its radius; $\therefore$ $4\pi r^2$ = its internal surface, and if s = the weight of an unit of volume of the fluid, the pressure

$$= s \,.\, r \,.\, 4\pi r^2 = 4\pi r^3 s;$$

$$\text{but the content of the sphere} = \tfrac{4}{3}\pi r^3,$$

$$\text{and its weight} = \tfrac{4}{3}\pi r^3 s;$$

$$\therefore \text{ the pressure : the weight of the fluid} :: 4\pi r^3 s : \tfrac{4}{3}\pi r^3 s$$
$$:: 3 : 1.$$

* $62\frac{1}{2}$ lbs. is the weight of a cubic foot of water.

A tetrahedron is filled with fluid. Compare the pressure on the base with the pressure on the sides, and the weight of the fluid.

If B and P represent the base and perpendicular altitude, the pressure on the base : that on a side :: $B \times P$: side $\times \frac{2}{3}P$
$:: 3 : 2$;

therefore the pressure on the three sides is double the pressure on the base. Also

the pressure on the base : the weight of the fluid
:: $B \times P$: solid content of the fluid
:: $3 : 1$;

therefore the whole pressure on the sides, the pressure on the base, and the weight of the fluid, are as 6, 3, 1.

To compare the pressure against the bottom of a cylinder with that against the sides.

If a = the altitude of the cylinder, and r = the radius of the base, the pressure on the bottom $= \pi a r^2 s$, and the pressure on the sides $= 2\pi r \cdot \frac{a}{2} \cdot as = \pi r a^2 s$;

∴ the pressure on [illegible] :: $\pi a r^2 s$: $\pi r a^2 s$ :: $r : a$.

[illegible] the whole pressure sustained by the cylinder

[illegible]

The centre of pressure of a plane surface immersed in a fluid is that point to which if a force equal to the whole force were applied in an opposite direction, it would keep the surface at rest. It will be shown afterwards to [illegible]

To find the pressure on the hinges of a flood-gate.

Let AD be the length of the gate, B and C the hinges, and P the centre of pressure; then the pressure at $C \times BC$ = pressure at $P \times BP$;

∴ pressure on the hinge $C = \dfrac{\text{pressure at } P \times BP}{BC}$.

Ex.—The length or depth of a flood-gate is 12 feet, breadth 14 feet; the hinges being placed one foot from the upper and lower extremity respectively. Find the pressure on each hinge.

The pressure of the water on each half of the gate is

$$= 7 \times 12 \times \frac{12}{2} \times 62.5 = 31500 \text{ lbs.}$$

Now, AD being the depth, and P the centre of pressure,

$$DP = \tfrac{1}{3}.AD = \frac{12}{3} = 4, \text{ and } BC = AD - AB - CD$$
$$= 12 - 2 = 10;\ BP = BD - DP = 11 - 4 = 7;$$

$$\therefore \text{ pressure on the hinge } C = \frac{31500 \times 7}{10} = 22050 \text{ lbs.};$$

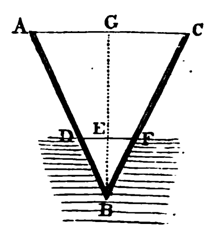
$$\therefore \text{ pressure on } B = 31500 - 22050 = 9450 \text{ lbs.};$$

Or thus;

$$\text{Pressure on } B = \frac{P \times PC}{BC} = \frac{31500 \times 3}{10} = 9450 \text{ lbs.};$$

$$\text{since } PC = PD - CD = 4 - 1 = 3.$$

ABC is a transverse section of an iron vessel $\frac{1}{4}$ of an inch thick, 40 feet in length, and of a form such that AC the breadth of beam is to BG as 4 is to 5, BG being 20 feet. Required the depth to which the vessel will sink, there being no deck.



$$AC : BG :: 4 : 5;$$

$$\therefore AC = \frac{20 \times 4}{5} = 16 \text{ feet.}$$

$$AB = \sqrt{BG^2 + AG^2}$$
$$= \sqrt{20^2 + \left(\frac{AC}{2}\right)^2}$$
$$= \sqrt{20^2 + 8^2} = \sqrt{464} = 21.54.$$

$$\text{Mean thickness of iron plates, \&c.} = \frac{1}{4} \text{ inch} = \frac{1}{48} \text{ foot}$$
$$= .0208 \text{ foot.}$$

Cubic feet of iron in vessel

$$= (2AB \times 40 + 2\frac{AC}{2} \times BG) \times .0208$$

$$= (2 \times 21.54 \times 40 + 16 \times 20) \times .0208$$

$$= 42.49856 \text{ cubic feet;}$$

$\therefore$ weight of vessel (at $487\frac{1}{2}$ lbs. per cubic foot)

$$= 42.49856 \times 487.5 \text{ lbs.} = 20718.09 \text{ lbs.;}$$

but weight of water displaced = weight of vessel

$$= 20718.09 \text{ lbs.;}$$

$$\therefore \text{ volume of water displaced} = \frac{20718.09}{62.5} \text{ cubic feet}$$

$$= 331.49 \text{ cubic feet.}$$

Now, if x = depth BE to which the vessel will sink in order to displace this volume of water, and if $DF = 2y$,

then volume $= x \times y \times 40 = 331.49$;

$$\therefore xy = \frac{331.49}{40} = 8.28975;$$

and $2y : x :: 4 : 5$; $\therefore 2x = 5y$; $\therefore y = \frac{2}{5}x = .4x$.

Substituting for y its value in terms of x,

$$x \times .4x = 8.28975;$$

$$x^2 = \frac{8.28975}{.4} = 20.724375;$$

$\therefore x = \sqrt{20.724375} = 4.5524$ feet = depth to which the vessel will sink.

A wall of masonry, each cubic foot of which weighs 100 lbs., the height is 10 feet, the thickness 3 feet. Find the height of water, so that the wall may be on the point of overturning.

$$10 \times 3 \times 1 \times 100 = 3000 = \text{weight of the wall;}$$

$$\text{the moment of the wall} = 3000 \times \frac{3}{2} = 4500.$$

Let x = height of the water;

$$\text{then } x \times \frac{x}{2} \times \frac{x}{3} \times 62.5 = \text{moment of the water};$$

$$\frac{62.5\,x^3}{6} = 4500;$$

$$x^3 = 432;$$

$$x = 7.5.$$

If the height of the wall be 8 feet, the thickness 6 feet, and the weight of a cubic foot 180 lbs. Find whether it will stand or fall, supposing the water level with the top.

The surface upon which the water presses is 8×1;

$$\therefore\ 8 \times 1 \times \frac{8}{2} \times 62.5 = 2000 = \text{weight of water};$$

$$2000 \times \frac{8}{3} = 5333\tfrac{1}{3} = \text{moment of the water};$$

$$8 \times 1 \times 6 \times 180 = 8640 \text{ lbs.} = \text{weight of the wall};$$

$$\therefore\ 8640 \times \frac{6}{2} = 25920 = \text{the moment of the wall.}$$

The moment of the wall being greater than that of the water, the wall will stand.

Remark.—It is only necessary to calculate for a wall one foot in length, for every foot will be equally stable.

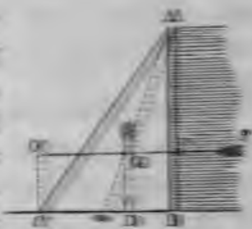
ON THE MODULUS OR CO-EFFICIENT OF STABILITY.

115. The Modulus of Stability* is the ratio of CI to CF (see figure, page 64); it is clear that a structure of any kind will be more stable the nearer the point I is to F, the point where the vertical through the centre of gravity meets the base. The celebrated Vauban generally made

* Professor Moseley uses the term *modulus of stability;* the French writers use *co-efficient* of stability.

$CI = \frac{4}{9} CE$. The modulus of stability of a wall or pier may be easily determined by the method of construction given at page 65.

There is an embankment ABC of brickwork, each cubic foot of which weighs 117 lbs.; AB is 14 feet, and BC is 6 feet. Find whether or not the embankment will be overthrown by the pressure of water on the surface AB; determining for a single foot of its length, since the embankment is uniform throughout its length, as also is the pressure upon it.



$$\text{Weight of embankment} = \frac{1 \times 14 \times 6}{2} \text{ cubic feet} \times 117 \text{ lbs.}$$
$$= 4914 \text{ lbs.} = W;$$

$$\text{pressure of water} = \left(1 \times 14 \times \frac{14}{2}\right) \text{ cubic feet} \times 62.5 \text{ lbs.}$$
$$= 6125 \text{ lbs.} = P;$$

Draw Am bisecting the base BC in m; take $mG = \frac{1}{3} Am$. Draw nGO through G parallel to BC; from G let fall the perpendicular GD, and at C erect a perpendicular CO; then n is at depth of $\frac{2}{3} AB$, being the centre of pressure through which P acts in direction nGO, perpendicular to the surface AB; also G is centre of gravity of the embankment, through which its weight W acts vertically in direction GD.

$$\therefore \text{ moment of embankment about } C = CD \times W$$
$$= 4 \times 4914 = 19656;$$

$$\text{and moment of pressure of water about } C = CO \times P$$
$$= Bn \times P = \frac{14}{3} \times 6125 = 28583\tfrac{1}{3};$$

therefore, since the latter exceeds the former, the embankment will be overturned.

ON THE HYDROSTATIC PARADOX.

116. The hydrostatic paradox may be explained upon the same principles as the mechanical powers; and an explanation conducted in this manner strips it of its paradoxical appearance.

The hydrostatic paradox is expressed thus:—A quantity of fluid, however small, may be made to balance a quantity, however large. Thus, let AB be a large vessel, and CD a small one which is connected with it. Then, if water be poured into either of them, it will stand at the same height in both; consequently there is an equilibrium between them.

Now, it must be observed that there is an equilibrium between them, in the same manner as in any of the mechanical powers. Take the lever, for instance: suppose 1 lb. to balance 100 lbs.; this is exactly the same as 1 lb. balancing 100 lbs. in the hydrostatic paradox. In the former case, the length of the arms of the lever must be in proportion to each other as 1 to 100; and, in the latter, the area of the vessels must be to each other as 1 to 100. But, properly speaking, 1 lb. does not, and cannot, balance 100 lbs. in any case whatever; for 1 lb. can only balance 1 lb. use what means you will.

Archimedes only required a fixed point to be able to sustain the whole earth; but, as Carnot very justly observed, if he had found it, it would not, in fact, have been Archimedes, but the fixed point or fulcrum, which would have sustained the earth.

In the hydrostatic paradox also, 1 lb. instead of balancing 100 lbs. only balances 1 lb. All the rest of the weight is supported by the vessel, in the same manner as the weight is supported on the fulcrum of the lever.

And, to show that the same takes place as in the lever, fix a piston which is water-tight into either of them,—suppose the greater, for instance; and if the area of the

If the specific gravity of the body and fluid are equal, then the body will remain at rest in any part of the fluid.

If the body be heavier than the fluid, it loses as much of its weight, when immersed, as is equal in weight to a quantity of the fluid of the same bulk.

If the specific gravity of the fluid be greater than that of the body, then the quantity of the fluid displaced by the part immersed is equal in weight to the weight of the whole body. Therefore, the specific gravity of the fluid is to that of the body, as the whole magnitude of the body is to the part immersed.

The specific gravities of equal solids are as their parts immersed in the same fluid.

The specific gravities of fluids are as the weights lost by the same immersed body.

When the Body is heavier than Water.

Rule.—Weigh it both in and out of water, and then say,
As the weight lost in water,
Is to the whole or absolute weight;
So is the specific gravity of water,
To the specific gravity of the body.

Ex.—Required the specific gravity of a piece of tin which weighs 23 lbs. but in water only 20 lbs.; the specific gravity of water being 1000.

$$23 - 20 = 3 \text{ lbs. weight lost in water.}$$
$$3 : 23 :: 1000 : 7333 = \text{the specific gravity.}$$

When the Body is lighter than Water.

Rule.—Attach to it a piece of another body, heavier than water, so that the mass compounded of the two may sink together. Weigh the denser body and the compound body separately, both out of the water and in it, and find how much each loses in the water by subtracting its weight in water from its weight in air; and subtract the less of these remainders from the greater. Then use the following proportion:

If it were required to know what weight must be added to it, so as to make the upper surface level with the water:—

The weight of the water displaced is equal to the weight of the body; but when the upper surface is level with the water, there is a cubic foot of water displaced, the weight of which is 1000 ounces. Therefore it will require as much weight to be added to it as to make the weight of the body 1000 ounces.

$1000 - 925 = 75$ ounces = the weight which must be added to it, to make the upper surface level with the surface of the water.

Ex. 2.—Suppose, by measurement, it be found that a man-of-war, with its ordnance, rigging, and appointments, sinks so deep as to displace 1300 tons of sea-water; what is the whole weight of the ship, supposing a cubic inch of sea-water to weigh .5949 of an ounce avoirdupois?

The weight of the water displaced is equal to the weight of the ship.

$$216 \text{ gallons} = 1 \text{ ton}.$$

$1300 \times 216 = 280800$ gallons; and if we take 277.2738 cubic inches to the gallon, then $280800 \times 277.2738 = 77858483.04$ cubic inches, and this multiplied by .5949 gives 46318011.5367 ounces = 1292.35 tons, the weight of the ship.

Ex. 3.—An irregular fragment of glass, in the scale, weighs 171 grains, and another of magnet 102 grains; but in water the first fetches up no more than 120 grains, and the other 79; what then will their specific gravities turn out to be?

The specific gravities of different bodies in the same fluid are as $\frac{w}{w'}$; w being the weight of the body, and w' the weight lost in the fluid.

$$\frac{171}{51} : \frac{102}{23} :: 3933 : 5202.$$

rkmen thought that substituting part silver was only a per perquisite; which taking air, Archimedes was pointed to examine it; who, on putting it into a vessel water, found it raised the fluid 8.2245 cubic inches; d having discovered that the inch of gold more critically ighed 10.36 ounces, and that of silver but 5.85 ounces, found by calculation what part of the king's gold had en changed. And you are desired to repeat the process.

10.36 : 1 :: 63 : 6.081 cubic inches.
5.85 : 1 :: 63 : 10.77 cubic inches.

By Alligation:

$$8.2245 \begin{cases} 6.081 \\ 10.77 \end{cases} \begin{matrix} 2.5455 \text{ cubic inches gold.} \\ 2.1435 \text{ cubic inches silver.} \end{matrix}$$

$$2.5455 + 2.1435 = 4.689;$$

Then, 4.689 : 63 :: 2.5455 : 34.2 ounces gold.
4.689 : 63 :: 2.1435 : 28.8 ounces silver.

Ex. 7.—What must be the thickness of a right-angled ne of copper, the inner diameter of which is 20 inches, that it may just float with its edge level with the surce of the fluid; the specific gravities of the copper and iid being as 9 to 1, and the interior and exterior surfaces ving a common base?

Let r = inner radius, t = thickness, and p = 3.1416. hen, since the cone is right-angled, the radius of the base equal to the altitude; therefore,

$$\frac{p\,(r+t)^2 \times (r+t)}{3} = 9.\ \frac{p\,[(r+t)^3 - r^3]}{3}$$

$$\therefore (r+t)^3 = 9\,(r+t)^3 - 9\,r^3.$$

$$8\,(r+t)^3 = 9\,r^3.$$

$$2\,(r+t) = r\,\sqrt[3]{9}.$$

LIQUIDS.

Sulphuric Acid	1,841	Burgundy Wine	991
Nitrous Acid	1,550	Olive Oil	915
Water from the Dead Sea	1,240	Muriatic Ether	874
Nitric Acid	1,218	Oil of Turpentine	870
Sea-Water	1,026	Liquid Bitumen	848
Milk	1,030	Alcohol, absolute	792
Distilled Water	1,000	Sulphuric Ether	716
Wine of Bourdeaux	994	Air at the Earth's Surface, about	$1\frac{2}{7}$

*** Since a cubic foot of water, at the temperature 40° Fahrenheit, weighs 1000 oz. avoirdupois, or 62½ lbs., the numbers in the preceding Tables exhibit very nearly the respective weights of a cubic foot of the several substances tabulated.—*Dr. Gregory's Edition of Dr. Hutton's Course.*

ON REVETEMENTS.

118. Coulomb was the first to form the happy idea of the prism of greatest pressure, which he gave in the *Savans Etrangers*, for 1773. Since the pressure of the earth against a revetement wall will vary as we vary the angle ω, it is necessary to determine that value which must be given to ω, or HbX, so that the line bX may cut off the prism that will press with the greatest force against the wall Hb; for, this being determined, a wall that will support this prism, will, *à fortiori*, support any other prism. Coulomb gave it for a vertical wall, and observes: "To have the pressure of a surface of earth against a vertical plane, it is necessary to find among all the surfaces that which, solicited by its weight, and retained by its friction and its cohesion, requires for its equilibrium to be supported by a maximum horizontal force; for it is evident that every other figure will require a less horizontal force to produce an equilibrium. As experience shows that the rupture of earth takes place nearly in a right line, it is sufficient in practice to find in an indefinite surface among all the prisms which press against a vertical plane, that which requires to

$$*P \left\{ \sin.\varepsilon \cos.a \sin.(\omega-\varepsilon) - \cos.(\omega-\varepsilon) \sin.a \sin.\varepsilon - \cos.\varepsilon \right.$$
$$\left. \sin.a \sin.(\omega-\varepsilon) - \cos.\varepsilon \,.\, \cos.a \,.\, \cos.(\omega-\varepsilon) \right\}$$
$$= \frac{p h^2 \sin.\omega \sin.(\omega-a-\varepsilon)}{2\cos.\varepsilon \,.\, \cos.(\omega-\varepsilon)} + \frac{ch \sin.a}{\cos.(\omega-\varepsilon)};$$

---

* The left hand side of the equation is the same as

$$\sin.\epsilon \left\{ \cos.a \sin.(\omega-\epsilon) - \sin.a \cos.(\omega-\epsilon) \right\}$$
$$- \cos.\epsilon \left\{ \cos.a \cos.(\omega-\epsilon) + \sin.a \sin.(\omega-\epsilon) \right\}$$
$$= \sin.\epsilon \,.\, \sin.(\omega-\epsilon-a) - \cos.\epsilon \,.\, \cos.(\omega-\epsilon-a)$$
$$= - \left\{ \cos.\epsilon \,.\, \cos.(\omega-\epsilon-a) - \sin.\epsilon \,.\, \sin.(\omega-\epsilon-a) \right\}$$
$$= - \cos.(\omega-\epsilon-a+\epsilon) = - \cos.(\omega-a).$$

The right hand side of the equation

$$= \frac{p h^2 \sin.\omega \,.\, \sin.(\omega-a-\epsilon) + 2ch \sin.a \cos.\epsilon}{2\cos.\epsilon \cos.(\omega-\epsilon)}.$$

Now, by page 29 Hall's Trigonometry, we have

$$2\sin.A \sin.B = \cos.(A-B) - \cos.(A+B)$$
$$= - \left\{ \cos.(A+B) - \cos.(A-B) \right\};$$
$$\therefore\ 2\sin.\omega \,.\, \sin.(\omega-a-\epsilon) = - \left\{ \cos.(\omega+\omega-a-\epsilon) - \cos.(\omega-\omega+a+\epsilon) \right\}$$
$$= - \left\{ \cos.(2\omega-a-\epsilon) - \cos.(a+\epsilon) \right\};$$
$$\therefore\ \sin.\omega \sin.(\omega-a-\epsilon) = - \left(\frac{\cos.(2\omega-a-\epsilon) - \cos.(a+\epsilon)}{2} \right);$$
$$\therefore\ -P\cos.(\omega-a) = \frac{-ph^2 \left\{ \cos.(2\omega-a-\epsilon) - \cos.(a+\epsilon) \right\} + 4ch \sin.a \cos.\epsilon}{4\cos.\epsilon \,.\, \cos.(\omega-\epsilon)};$$
$$\therefore\ P = \frac{ph^2 \left\{ \cos.(2\omega-a-\epsilon) - \cos.(a-\epsilon) \right\} - 4ch \sin.a \cos.\epsilon}{4\cos.\epsilon \cos.(\omega-\epsilon) \,.\, \cos.(\omega-a)};$$

$$\text{Also, } \cos.A \cos.B = \tfrac{1}{2} \left\{ \cos.(A+B) + \cos.(A-B) \right\};$$
$$\therefore\ \cos.(\omega-\epsilon)\cos.(\omega-a) = \tfrac{1}{2} \left\{ \cos.(\omega-\epsilon+\omega-a) + \cos.(\omega-\epsilon-\omega+a) \right\}$$
$$= \tfrac{1}{2} \left\{ \cos.(2\omega-a-\epsilon) + \cos.(a-\epsilon) \right\};$$

hence, by substitution, we have

$$P = \frac{p h^2 \cos.(2\omega-a-\epsilon) - p h^2 \cos.(a-\epsilon) - 4ch \sin.a \cos.\epsilon}{2\cos.\epsilon \,.\, \left\{ \cos.(2\omega-a-\epsilon) + \cos.(a-\epsilon) \right\}},$$

which is to be a max.;

$$\text{or } \frac{p h^2 \cos.(2\omega-a-\epsilon) - p h^2 \cos.(a-\epsilon) - 4ch \sin.a \sin.\epsilon}{\cos.(2\omega-a-\epsilon + \cos.(a-\epsilon)}$$

$= \text{max.}$, since $2\cos.\epsilon$ is constant.

or, by the analysis of M. Persy, (see note, page 212,)

$$P \cos.(\omega - a)$$
$$= \frac{ph^2 \cos.(2\omega - a - \varepsilon) - ph^2 \cos.(a + \varepsilon) - 4ch \sin.a \cos.\varepsilon}{4 \cos.\varepsilon \,.\, \cos.(\omega - \varepsilon)}$$

and, finally,

$$P = \frac{ph^2 \cos.(2\omega - a - \varepsilon) - ph^2 \cos.(a + \varepsilon) - 4ch \sin.a \cos.\varepsilon}{2 \cos.\left\{\cos.(2\omega - a - \varepsilon) + \cos.(a - \varepsilon)\right\}} \quad (5.)$$

This expression must be a maximum, or, since $2 \cos.\varepsilon$ is a constant quantity,

$$P = \frac{ph^2 \cos.(2\omega - a - \varepsilon) - ph^2 \cos.(a + \varepsilon) - 4ch \sin.a \cos.\varepsilon}{\cos.(2\omega - a - \varepsilon) + \cos.(a - \varepsilon)};$$

as ω is the only variable, the above expression may be put into the following form:—

$$P = \frac{a \cos.(u + b) - g}{\cos.(u + b) + f};$$

$$\frac{dP}{du} = \frac{-a \sin.(u+b)\left\{\cos.(u+b) + f\right\} + \sin.(u+b)\left\{a \cos.(u+b) - g\right\}}{\left\{\cos.(u+b) + f\right\}^2}$$

$$= \frac{-\sin.(u+b)}{\left\{\cos.(u+b) + f\right\}^2}\left\{a \cos.(u+b) + af - a \cos.(u+b) + g\right\}$$

$$= -\frac{\sin.(u+b)}{\left\{\cos.(u+b) + f\right\}^2}(af + g) = 0;$$

$$\therefore u + b = 0, \text{ or } 2\omega - a - \varepsilon = 0;$$

$$\therefore \omega = \tfrac{1}{2}(a + \varepsilon).$$

The second differential co-efficient—

$$\frac{d^2P}{du^2} = -\frac{\cos.^2(u+b) + 2 \sin.^2(u+b) + f \cos.(u+b)}{\left\{\cos.(u+b) + f\right\}^3}(af + g);$$

is evidently negative when $\dfrac{dP}{dx} = 0$;

hence P is a maximum when $\omega = \tfrac{1}{2}(a + \varepsilon)$.

The prism of greatest pressure is therefore found by

bisecting the angle formed by the revetement-wall and the natural slope.

Since $2\omega - a - \varepsilon = 0$, $\cos.(2\omega - a - \varepsilon) = 1$;

substituting this in (5), we have

$$P = \frac{ph^2\{1-\cos.(a+\varepsilon)\} - 4ch\sin.a\cos.\varepsilon}{2\cos.\varepsilon.\{1+\cos.(a-\varepsilon)\}}$$

$$= \frac{\frac{ph^2}{\cos.\varepsilon}.\sin.^2\frac{1}{2}(a+\varepsilon) - 2ch\sin.a}{2\cos.^2\frac{1}{2}(a-\varepsilon)} \quad \ldots\ldots (6.)$$

This expression is $= 0$, when $h = 0$; and when h increases, it is at first negative, and then comes back to nothing again; let h' be the value at that time, then, putting the value of P in equation (6) $= 0$, we have

$$h' = \frac{2c\sin.a\cos.\varepsilon}{p\sin.^2\frac{1}{2}(a+\varepsilon)};$$

$$\therefore\ 2c\sin.a = \frac{ph'\sin.^2\frac{1}{2}(a+\varepsilon)}{\cos.\varepsilon};$$

this substituted (6) gives

$$P = \frac{\frac{ph^2}{\cos.\varepsilon}\sin.^2\frac{1}{2}(a+\varepsilon) - \frac{phh'\sin.^2\frac{1}{2}(a+\varepsilon)}{\cos.\varepsilon}}{2\cos.^2\frac{1}{2}(a-\varepsilon)}$$

$$= \frac{1}{2}\frac{ph}{\cos.\varepsilon}(h-h')\frac{\sin.^2\frac{1}{2}(a+\varepsilon)}{\cos.^2\frac{1}{2}(a-\varepsilon)}$$

$$= \frac{1}{2}ph(h-h')\frac{\sin.^2\frac{1}{2}(a+\varepsilon)\cos.\varepsilon}{\cos.^2\varepsilon.\cos.^2\frac{1}{2}(a-\varepsilon)}.$$

Navier puts $\dfrac{\sin.^2\frac{1}{2}(a+\varepsilon)}{\cos.^2\varepsilon\cos.^2\frac{1}{2}(a-\varepsilon)} = t^2$, or

$$\frac{\sin.\frac{1}{2}(a+\varepsilon)}{\cos.\varepsilon.\cos.\frac{1}{2}(a-\varepsilon)} = t;\text{*}$$

$$\therefore\ P = \frac{1}{2}ph(h-h')t^2\cos.\varepsilon \quad \ldots\ldots (7.)$$

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* This is the ratio of the base of the prism of greatest pressure to its height, for, by note, page 210,

$$\frac{HX}{h} = \frac{\sin.\omega}{\cos.\varepsilon\cos.(\omega-\varepsilon)} = \frac{\sin.\frac{1}{2}(a+\varepsilon)}{\cos.\varepsilon.\cos.\frac{1}{2}(a-\varepsilon)};$$

since $\omega = \frac{1}{2}(a+\varepsilon)$.

We may now proceed to find the effect of P on the plane Hb, and must therefore find its point of application.

In equation (7) let z be any variable distance from B, then

$$P = \tfrac{1}{2} p\, t^2 \cos.\varepsilon\; z\,(z - h');$$

this differentiated

$$dP = p t^2 \cos.\varepsilon\,.\,(z - \tfrac{1}{2} h')\, dz.$$

This represents the pressure on an element of Hb; let P be the point where this pressure acts;

$$\text{then } bP \cos.\varepsilon = h - z;$$

$$\therefore\; bP = \frac{h-z}{\cos.\varepsilon};$$

Now, measuring the moments from b, the moment of this element is

$$dP \times bP = p t^2 \cos.\varepsilon\,(z - \tfrac{1}{2} h')\, dz\,.\,\frac{h-z}{\cos.\varepsilon}$$

$$= p t^2\,(z - \tfrac{1}{2} h')\, dz\,(h - z).$$

Now, the sum of the moments of all the pressures dP will give the moment of the pressure P; or, which is the same,

$$p t^2 \int_o^h (z - \tfrac{1}{2} h')\,(h - z)\, dz$$

$$= p t^2 \left(\frac{h^3}{2} - \frac{h^3}{3} + \tfrac{1}{2}\frac{h^2 h'}{2} - \tfrac{1}{2} h^2 h'\right)$$

$$= \tfrac{1}{2} p t^2 h^2 \,(\tfrac{1}{3} h - \tfrac{1}{2} h');$$

$$\therefore\; bP = \frac{\tfrac{1}{2} p t^2 h^2\,(\tfrac{1}{3} h - \tfrac{1}{2} h')}{\tfrac{1}{2} p h\,(h - h')\, t^2 \cos.\varepsilon} = \frac{h\,(\tfrac{1}{3} h - \tfrac{1}{2} h')}{(h - h')\cos.\varepsilon};$$

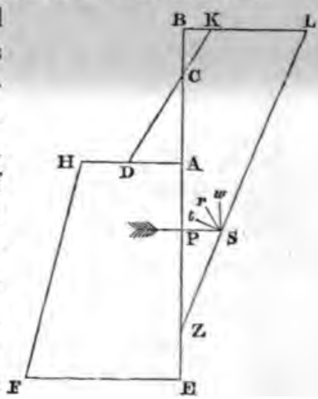
which is the distance from b, where the pressure must be applied.

If we want to find the point of application of the pressure of water against an upright wall, we must make

$k' = 0$, since the cohesion is nothing; and $\varepsilon = 0$, therefore $\cos.\varepsilon = 1$; then $bP = \frac{1}{3}h$, the same as observed at p. 196 for the distance of the centre of pressure.

ON SURCHARGED REVETEMENTS.

119. Poncelet, in the *Mémorial de l'Officier du Génie,* No. 13, gives a general solution to this case. Professor Moseley, in his great work, (*Mechanical Principles of Engineering and Architecture,*) also gives a neat solution to the same. The latter philosopher has found the pressure in a manner equally simple and elegant, and the following solution is nearly the same as he has given. In finding the moment of the thrust, however, Moseley neglects the influence of the small part DCA on the equilibrium, and integrates between the limits of BA and BZ, whereas Poncelet integrates between the limits of BE and BC.



Let DK represent the natural slope of the earth, and KL its horizontal surface; EA the internal face of the wall. Produce EA and LK to meet in B; let AZ represent any portion of internal surface of the wall, and $KCZL$ the mass of earth whose pressure is sustained by that portion of the surface of the wall.

Let $\angle BZL = i$, $BZ = z$, $BC = C$, p = weight of a cubic foot of earth.

Also, since DK is the natural slope of the earth, if ϕ represent the limiting angle of resistance, then the inclination of DK to the horizon is represented by ϕ.

Now, if W represent the weight of the mass $CZLK$, then, since P in the direction PS, W in the direction wS, and the resistance r in the direction rS, are pressures in [illegible] (where rS is inclined to the perpendicular St,

at an angle equal to the limiting angle of resistance,) then, by Art. 3, Cor. 1,

$$P : W :: \sin. \omega sr : \sin. Psr;$$

$$\omega sr = \omega st - tsr = 90 - \iota - \phi;$$

$$Psr = Pst + tsr = \iota + \phi;$$

$$P = W . \frac{\sin. \omega sr}{\sin. Psr} = W \frac{\sin.\{90-(\iota+\phi)\}}{\sin. (\iota+\phi)} = W . \cot. (\iota+\phi).$$

Also, $W = p\ (CZLK) = p\{BZL - BCK\}$

$$= p\left\{\tfrac{1}{2}(BZ . BL - BC . BK)\right\}$$

$$= \tfrac{1}{2} p \{ z^2 \tan. i - c^2 \cot. \phi \};$$

$$\therefore P = \tfrac{1}{2} p \left\{ \frac{z^2 \tan. i - c^2 \cot. \phi}{\tan. \overline{i + \phi}} \right\} \quad . \ . \ . \ . \quad (2.)$$

Now, since P is a maximum function of (i), if therefore

$$u = \frac{z^2 \tan. i - c^2 \cot. \phi}{\tan. \overline{i + \phi}},$$

then $\frac{du}{di} = 0$, and $\frac{d^2 u}{d^2 i}$ is negative.

Differentiating we have

$$\frac{du}{di} = \frac{z^2 \sec.^2 i \tan. \overline{i + \phi} - (z^2 \tan. i - c^2 \cot. \phi) \sec.^2 \overline{i + \phi}}{\tan. \overline{i + \phi}},$$

{ and multiplying both numerator and denominator by $\cos.^2 i \cos.^2 \overline{i + \phi}$ }

$$\frac{du}{di} = \frac{z^2 \sin. \overline{i+\phi} . \cos. \overline{i+\phi} - z^2 \sin. i \cos. i + c^2 \cot. \phi . \cos.^2 i}{\sin.^2 \overline{i+\phi} \cos.^2 i} = 0;$$

$$\therefore z^2 \left\{ \tfrac{1}{2} \sin. 2 \overline{i+\phi} - \sin. i \cos. i \right\} + c^2 \cot. \phi \cos.^2 i = 0$$

$$= \frac{z^2 \left\{ \tfrac{1}{2} \sin. 2(\iota+\phi) - \sin. \iota \cos. \iota \right\}}{\cos.^2 \iota} + c^2 \cot. \phi = 0;$$

$$\text{or } \tfrac{1}{2} \frac{\sin. 2(\iota + \phi) - \sin. \iota \cos. \iota}{\cos.^2 \iota} + \frac{c^2}{z^2} \cot. \phi = 0;$$

$$\frac{1}{2}\,\frac{\{\sin.2\iota\,\cos.2\phi + \cos.2\iota\,\sin.2\phi\}}{\cos.^2\iota} - \tan.\iota + \frac{r^2}{z^2}\cot.\phi = 0;$$

$$\frac{1}{2}\,\frac{\{2\sin.\iota\,\cos.\iota\,\cos.2\phi + (2\cos.^2\iota - 1)\sin.2\phi\}}{\cos.^2\iota}$$

$$- \tan.\iota + \frac{r^2}{z^2}\cot.\phi = 0;$$

$$\frac{1}{2}\{2\tan.\iota\,\cos.2\phi + (2 - \sec.^2\iota)\sin.2\phi\}$$

$$- \tan.\iota + \frac{r^2}{z^2}\cot.\phi = 0;$$

$$\tan.\iota\,\cos.2\phi + \frac{(1 - \tan.^2\iota)\sin.2\phi}{2} - \tan.\iota + \frac{r^2}{z^2}\cot.\phi = 0;$$

$$-\tan.\iota(1 - \cos.2\phi) + (1 - \tan.^2\iota)\sin.\phi\,\cos.\phi + \frac{r^2}{z^2}\cot.\phi = 0;$$

$$-2\tan.\iota\,\sin.^2\phi + (1 - \tan.^2\iota)\sin.\phi\,\cos.\phi + \frac{r^2}{z^2}\cot.\phi = 0;$$

$$-2\tan.\iota\,\tan.\phi + 1 - \tan.^2\iota + \frac{r^2}{z^2}\text{cosec.}^2\phi = 0;$$

$$\therefore \tan.^2\iota + 2\tan.\phi\,\tan.\iota = 1 + \frac{r^2}{z^2}\text{cosec.}^2\phi;$$

$$\tan.^2\iota + 2\tan.\iota\,\tan.\phi + \tan.^2\phi = 1 + \tan.^2\phi + \frac{r^2}{z^2}\text{cosec.}^2\phi$$

$$= \sec.^2\phi + \frac{r^2}{z^2}\text{cosec.}^2\phi;$$

$$\tan.\iota + \tan.\phi = \sqrt{\sec.^2\phi + \frac{r^2}{z^2}\text{cosec.}^2\phi};$$

$$\therefore \tan.\iota = \sqrt{\sec.^2\phi + \frac{r^2}{z^2}\text{cosec.}^2\phi} - \tan.\phi;$$

substituting this value of tan. ι in (P), observing that

$$\frac{1}{\tan.(\iota + \phi)} = \frac{1 - \tan.\iota\,.\,\tan.\phi}{\tan.\iota + \tan.\phi}.$$

Now, $1 - \tan.\iota\,.\,\tan.\phi$

$$= 1 - \tan.\phi\sqrt{\sec.^2\phi + \frac{r^2}{z^2}\text{cosec.}^2\phi} + \tan.^2\phi$$

$$= \sec.^2\phi - \tan.\phi \sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi};$$

Also, $\tan.\iota + \tan.\phi = \sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}$;

$$\therefore \frac{1}{\tan.(\iota + \phi)} = \frac{\sec.^2\phi - \tan.\phi\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}}{\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}}$$

$$= \frac{\sec.^2\phi\left\{1 - \sin.\phi\cos.\phi\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}\right\}}{\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}};$$

And $z^2\tan.\iota - c^2\cot.\phi$

$$= z^2\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi} - z^2\tan.\phi - c^2\cot.\phi$$

$$= z^2\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}$$

$$- z^2\sin.\phi\cos.\phi\left\{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi\right\}$$

$$= z^2\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}$$

$$\left\{1 - \sin.\phi\cos.\phi\sqrt{\sec.^2\phi + \frac{c^2}{z^2}\operatorname{cosec.}^2\phi}\right\};$$

$$\therefore P = \tfrac{1}{2}p\left\{\frac{z^2\tan.\iota - c^2\cot.\phi}{\tan.(\iota + \phi)}\right\}$$

$$= \tfrac{1}{2}p\,z^2\sec.^2\phi\left\{1 - \sin.\phi\cos.\phi\sqrt{\sec.^2\phi + \frac{c^2}{z_2}\operatorname{cosec.}^2\phi}\right\}$$

$$= \tfrac{1}{2}p\,z^2\sec.^2\phi\left\{1 - \sqrt{\sin.^2\phi + \frac{c^2}{z^2}\cos.^2\phi}\right\}^2;$$

$$\frac{dP}{dz} = p\left\{z\sec.\phi - (z^2\tan.^2\phi + c^2)^{\frac{1}{2}}\right\}$$

$$\left\{\sec.\phi - \frac{z\tan.^2\phi}{(z^2\tan.^2\phi + c^2)^{\frac{1}{2}}}\right\}$$

$$= p\left\{ z \sec.^2\phi - \sec.\phi \cdot \frac{z^2 \tan.^2\phi}{(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}}} \right.$$
$$\left. - \sec.\phi\,(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}} + z \tan.^2\phi \right\}$$

$$= p\left\{ z(\sec.^2\phi + \tan.^2\phi) - \sec.\phi\left(\frac{z^2 \tan.^2\phi + c^2 - c^2}{(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}}}\right) \right.$$
$$\left. - \sec.\phi\,(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}} \right\}$$

$$= p\left\{ z(\sec.^2\phi + \tan.^4\phi) - \sec.\phi\,(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}} \right.$$
$$\left. - \frac{c^2 \sec.\phi}{(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}}} - \sec.\phi\,(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}} \right\}$$

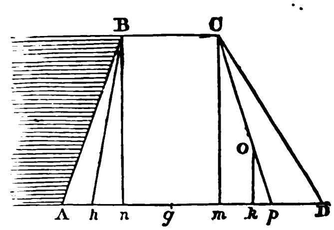
$$= p\left\{ z(\sec.^2\phi + \tan.^2\phi) - 2\sec.\phi\,(z^2 \tan.^2\phi + c^2)^{\frac{1}{2}} \right.$$
$$\left. - \frac{c^2 \sec.\phi}{(z^2 \tan.^2\phi - c^2)^{\frac{1}{2}}} \right. .$$

Hence the moment of the thrust

$$\int \frac{dP}{dz} z = p \int_c^z \left\{ z^2(\sec.^2\phi + \tan.^2\phi) - 2z\sec.\phi\,(z^2\tan.^2\phi + c^2)^{\frac{1}{2}} \right.$$
$$\left. - \frac{c^2 z \,.\, \sec.\phi}{(z^2\tan.^2\phi + c^2)^{\frac{1}{2}}} \right\}$$

$$= p\left\{ \begin{array}{l} \frac{1}{3}(z^3 - c^3)(\sec.^2\phi + \tan.^2\phi) - \frac{2}{3}\sec.\phi\cot.^2\phi \\ \qquad\qquad \left\{ z^2 \tan.^2\phi + c^2)^{\frac{3}{2}} - (c^2 \tan.^2\phi + c^2)^{\frac{3}{2}} \right\} \\ - c^2 \sec.\phi \cot.^2\phi \left\{ z^2 \tan.^2\phi + c^2)^{\frac{1}{2}} - (c^2 \tan.^2\phi + c^2)^{\frac{1}{2}} \right\} \end{array} \right\}$$

To find the pressure of water against a dyke, as in the accompanying figure.



Let H = the height, l = length, and b = breadth of the dyke, w = weight of a cubic foot of water, W = weight of a cubic foot of masonry, a = the angle BAn, and $\beta = CDm$; $e = BC$, the thickness of the dyke at the top, and R = the pressure of the water.

$$\text{Area } CDm = \tfrac{1}{2}\, mD\,.\,Cm, \text{ but } mD = \frac{H}{\tan.\beta};$$

$$\therefore \tfrac{1}{2}\,\frac{H}{\tan.\beta}\,.\,H = \frac{H^2}{2\tan.\beta};$$

$$\text{and the weight} = \frac{H^2}{2\tan.\beta} \times l \times W.$$

In the same manner the weight of $ABn = \dfrac{H^2}{2\tan.a} \times l\,W$;

$$\text{and the weight of } BCmn = H \times e \times l \times W.$$

Now, since 0 is the centre of gravity of the triangle CDm, we have

$$mk = \tfrac{1}{3}\,mD, \text{ or } Dk = \tfrac{2}{3}\,mD = \tfrac{2}{3}\,.\,\frac{H}{\tan.\beta};$$

$$\therefore \text{ the moment of } CDm = \tfrac{2}{3}\,.\,\frac{H}{\tan.\beta} \times \frac{H^2}{2\tan.\beta} \times l \times W.$$

The moment of $BCnm = e \times H \times l \times W\left(\dfrac{e}{2} + \dfrac{H}{\tan.\beta}\right)$;

the moment of ABn

$$= \frac{H}{2\tan.a} \times W \times l\left(\frac{H}{3\tan.a} + e + \frac{H}{\tan.\beta}\right);$$

$\therefore$ the whole moment of the prism is

$$Wl\left\{\frac{H^2}{2\tan.\beta} \times \frac{2H}{3\tan.\beta} + eH\left(\tfrac{1}{2}\,e + \frac{H}{\tan.\beta}\right)\right.$$
$$\left. + \frac{H^2}{2\tan.a}\left(\frac{H}{3\tan.a} + e + \frac{H}{\tan.\beta}\right)\right\} \quad \ldots \text{(1.)}$$

The pressure R of the water acts perpendicularly on the face AB, and is $= l \times b \times w \times \tfrac{1}{2}\,H$. Resolve this force into two others; one horizontal, which tends to overturn the dyke, and the other vertical, which tends to prevent its overturning; the horizontal force $= R\sin.a$, and its moment with respect to the point $D = R\sin.a \times \tfrac{1}{3}\,H$; the vertical force $= R\cos.a$, and its moment

$$= R\cos.\alpha \times \left(\frac{H}{\tan.\beta} + \frac{2H}{3\tan.\alpha} + e\right).$$

Hence, by the equality of moments, we have

$$R\sin.\alpha \times \tfrac{1}{3}H = W \ldots \frac{H^2}{2\tan.\beta} \times \frac{2}{3}\frac{H}{\tan.\beta}$$

$$+ eH\left(\tfrac{1}{2}e + \frac{H}{\tan.\beta}\right) + \frac{H^2}{2\tan.\alpha}\left(\frac{H}{3\tan.\alpha} + e + \frac{H}{\tan}\right.$$

$$+ R\cos.\alpha\left(\frac{H}{\tan.\beta} + \frac{2H}{3\tan.\alpha} + e\right) \quad \ldots\ldots \quad (2.)$$

When the faces are equally inclined $\alpha = \beta$; hence

$$R\sin.\alpha \times \tfrac{1}{3}H = W\left(eH + \frac{H^2}{\tan.\alpha}\right).\left(\tfrac{1}{2}e + \frac{H}{\tan.}\right.$$

$$+ R\cos.\alpha\left(e + \frac{5H}{3\tan.\alpha}\right) \quad \ldots\ldots\ldots \quad (3.)$$

If the faces are vertical, then $\alpha = 90^{\circ}$, and then $\sin.\alpha = 1$, $\cos.\alpha = 0$, $\tan.\alpha = \infty$. Substitute

$$R \ldots H = W \ldots \tfrac{1}{2}e^2 H:$$

$$\ldots H = l \,.\, w \,.\, \frac{H^2}{2}, \text{ because wher}$$

$$\ldots H;$$

$$\ldots \frac{H}{3} = \frac{W \,.\, l \,.\, e^2 \,.\, H}{2};$$

$$\ldots \frac{H^2}{3} = W \,.\, e^2;$$

$$= \frac{w}{3W} \,.\, H^2;$$

$$\ldots H\sqrt{\frac{w}{3W}} \quad \ldots\ldots \quad (4.)$$

$\ldots 0$, then the general equatior

$$R \sin. \alpha . \tfrac{1}{3} H = W \ [illegible]$$ [illegible]

or, since $R = l . b . w . \tfrac{1}{2} H$

$$l . b . w . \tfrac{1}{2} H . \tfrac{1}{3} H = W \ [illegible]$$

$$\frac{H^2 b w}{6} = W \ [illegible]$$

$$\frac{H^2 b . w}{6} = W \ [illegible]$$ [illegible]

As the length of the embankment does not enter these expressions, it is clear that we may use any unit of that kind which is most convenient, conformably to the remark made at page [illegible].

In all these formulae the calculation is for strict equilibrium, but this being known any required stability may be given.

The pressure of earth against such a wall or dyke may be easily determined, for Professor Moseley has shown, in his admirable work on Engineering and Architecture, that we may find the pressure of the earth by regarding it as a fluid, having the weight of a cubic foot equal to the weight of a cubic foot of the earth, multiplied by the square of the tangent of half the angle which the natural slope makes with the vertical.

HYDRAULICS.

Hydraulics, or Hydrodynamics, treats of the motion of fluids, and the forces with which they act upon bodies against which they strike, or which move in them.

The velocity with which a fluid issues from a very small orifice in the bottom or side of a vessel that is kept constantly full, is equal to that which a heavy body would acquire by falling from the level of the surface of the fluid to the level of the orifice.

Therefore, if h = height of the fluid above the orifice, g = the velocity acquired by a falling body in one second, and v = the velocity with which the water issues,

$$v = \sqrt{2gh}.$$

The quantity of water that issues in one second through a given orifice is equal to a column of water having the area of the orifice for its base, and the velocity with which the fluid issues for its altitude.

That is, if A = the area of the orifice, and Q = the quantity of fluid running out in one second,

$$Q = A\sqrt{2gh}.$$

Or, if Q and h be given, then $A = \dfrac{Q}{\sqrt{2gh}}$;

And if Q and A be given, $h = \dfrac{2gA^2}{Q^2}$.

Experiments do not exactly agree with this theory as to the quantity of water run out; for the vein of water that issues through the small orifice suffers a contraction, by which its section has been found to be diminished in the ratio of nearly 5 to 7. Therefore, instead of $Q = A\sqrt{2gh}$, we have $Q = \frac{5}{7}A\sqrt{2gh}$. But $\frac{5}{7}$ is nearly $= \dfrac{1}{\sqrt{2}}$;

therefore, we may take $Q = \frac{1}{\sqrt{2}} \cdot A\sqrt{2gh} = \frac{1}{\sqrt{2}} \times A \times \sqrt{2} \times \sqrt{gh} = A\sqrt{gh}$. But $\sqrt{gh}$ is the velocity which a heavy body acquires in falling through $\frac{h}{2}$; consequently, the velocity of the water at the orifice is found equal to that which a heavy body would acquire in falling freely through half the altitude.

The experiments of Bossut show that the actual discharge through a hole made in the side or bottom of a vessel, is to the theoretical as 1 to .62, or nearly as 8 to 5. Consequently, the theoretical discharge must be diminished in this ratio to have the true discharge. Also, if a pipe from 1 to 2 inches long be inserted in the aperture, the contraction of the vein is prevented, and the actual discharge is to the theoretical as 4 to 5.

The quantities discharged are as the square root of the depth multiplied into the areas of the orifices.

The following is from Mr. Banks's Treatise on Mills:—

When the water is discharged through perpendicular sections, the velocity at the bottom of the orifice is rather greater than that at the top; but if the depth and breadth of the orifice be but small compared with its depth below the surface of the dam, we may take the depth of the centre of the hole for the mean depth, without any sensible error.

Newton concluded that the real velocity, was less than the theoretical or computed velocity, in the ratio of 1 to $\sqrt{2}$; Abbé Bossut as 100 to 150; and Michelotti as 5 to 8. Mr. Banks gives the experiments of these and others as follow:—

Newton . . .	.707	—that is, $\frac{707}{1000}$ part of the computed velocity.
Bossut . . .	.615	
Banks . . .	.750	
Michelotti . .	.625	
Helsham . . .	.705	
Smeaton . . .	.631	
6)	4.033	
	.672	= mean.

The mean velocity of these experiments is [illegible] of the computed velocity, or their ratio is as $5\frac{3}{8}$ to 8, viz. at the depth of a foot the actual velocity from them is $5\frac{3}{8}$, or 5.4 feet, and in the same proportion for any other depth; for the computed velocity is $v = \sqrt{64\frac{1}{3}h} = 8\sqrt{h}$ nearly. Then, $.672 \times 8\sqrt{h} = 5.4\sqrt{h}$; and if $h = 1$ foot, it becomes 5.4. From this, Mr. Banks gives the following rule:—

Find the depth of the vessel in feet; then multiply the square root of that depth by 5.4, and the product will give the velocity in feet per second. This, multiplied by the area of the orifice in feet, gives the number of cubic feet which flows out in one second.

***Ex.*—If the orifice be made 9 feet below the surface of the water, the breadth of the orifice 3 feet, and length 4 inches, what quantity of water will flow out in one second?**

$\sqrt{9} = 3$, and $3 \times 5.4 = 16.2$ = the velocity per second.

4 inches is $\frac{1}{3}$ of a foot.

3 feet $\times \frac{1}{3} = 1$ foot, and $1 \times 16.2 = 16.2$ cubic feet, water which flows out in a second.

To find the quantity of water discharged through notches or slits cut in the side of a vessel, the surface being without motion, and if the velocity below varies as the square root of the depth, then the area of the perpendicular section will be a parabola, and may be found by the following rule:—

Rule.—Multiply the velocity at the bottom by the depth, and two-thirds of the product will give the area; and the area, multiplied by the breadth of the slit, gives the number of cubic feet discharged.

Ex.—If the depth of a notch or slit be 6 inches, and the breadth 8 inches, required the quantity which will be discharged in 20 seconds.

The depth in feet is .5, the square root of which is .707, $\therefore 5.4 \times .707 = 3.8178$, two-thirds of which is 2.5452.

By the rule, $2.5452 \times .5 + 1.2727$, which, multiplied by the breadth, which is $\frac{5}{6}$ of a foot, gives .8484 feet per second; and $.8484 \times 20 = 16.9680$ cubic feet.

The Rev. Morgan Cowie, M.A., Principal of the College of Civil Engineers, Putney, has a paper in the *Mechanic's Magazine* for February, 1848, entitled a "Review of the Evidence given before the Metropolitan Sanitary Commissioners on the application of Hydraulic Science to the improvement of the Sewerage and Drainage of Towns." In this paper he has made some very excellent observations—

"1. As to *gauging*.

"2. As to the *velocity* with which fluids will discharge themselves in pipes not kept constantly full, with different inclinations.

"3. As to the *capacity* necessary for receiving and carrying off united streams.

"4. As to the nature of the substance of the channel, what influence it has on velocity.

"1. *The gauging*.

"Complaints are made that the formulæ investigated by mathematicians and others do not give accurate results. I should ask, in reply:—1st, How the *accuracy is tested?* 2d, How was the experiment made for finding the data required in the formulæ? Mr. Phillips gives correctly the rules adopted (not from theory though by mathematicians, but from experiment) for determining the mean velocity when the surface velocity is known, except in the case of M. Prony's formulæ for finding the mean velocity (v) from the surface velocity (V),

$$v = V\,\frac{V + 2.37}{V + 3.15}\ \text{in metres;}$$

or in feet,

$$v = V\,\frac{V + 7.776}{V + 10.335};$$

The water brought from Scy to Metz, to supply the fountains, is conducted by pipes of 3000 meters in length, and .08 meter in diameter; the height of the reservoir is 20 meters.

Poncelet's formula is

$$v = 26.44 \sqrt{\frac{d \,.\, h}{l + 54 \,.\, d}};$$

where v = the velocity in meters, d = diameter in meters, and l = length in meters. This reduced to feet gives, since one meter is = 3.2808992 feet,

$$v = 26.44 \sqrt{\frac{d \times 3.2808992 \times h \times 3.2808992}{l \times 3.2808992 + 54 \times 3.2808992}}$$

$$= 26.44 \times \sqrt{3.2808992} \,.\, \sqrt{\frac{dh}{l + 54 \,.\, d}}$$

$$= 47.9 \sqrt{\frac{dh}{l + 54 \,.\, d}} = \text{velocity in feet.}$$

The co-efficient 47.95, given by Mr. Cowie, is rather too large; it is not quite 47.9, but 47.9 brings out the velocity in feet, corresponding to Poncelet's velocity in meters, extremely near.

Taking the above data, we have

l = 3000 meters = 3.2808992 × 3000 = 9842.6976 feet;

h = 20 meters = 3.2808992 × 20 = 65.617984 feet;

d = .08 meters = 3.2808992 × .08 = .262471936 feet;

$$v = 47.9 \sqrt{\frac{65 \,.\, 617984 \times .26247}{9842 \,.\, 6976 + 54 \times .26247}} = 47.9 \times .0418$$
$$= 2.00222 \text{ feet per second.}$$

Poncelet finds v = .608 meters; this reduced to feet is

$$3.2808992 \times .608 = 1.9947867136;$$

the above only differs from Poncelet's velocity by one seven-thousandth of a foot.

The area of the section of the pipe = .7854 × $(.26247)^2$ = .0541 square feet.

The quantity of water per second [illegible]
= .10832 cubic feet. [illegible]

∴ the supply of water [illegible]
= 9358.848 cubic feet, which [illegible]
cubic meters. [illegible]
the above is more correct [illegible]
to a greater number of places.

"2. As to the velocity.

"In an open canal [illegible]

[illegible]

"These constants are [illegible]
from experiments made [illegible]
considerable variation [illegible]
and this is the [illegible]
the common [illegible]
between the [illegible]
miles [illegible]

"The [illegible]

[illegible]

speaking of the velocity [illegible]

stant accessions, moving with different velocities, Mr. Cowie observes—

"I would calculate in the following way:—Suppose a sewer five miles long, with a fall from one end to the other of 200 feet, *i. e.* a fall of 40 feet per mile, and suppose it to receive additions of water at every half-mile by sewers of given dimensions, the water coming down these sewers will have fallen through 20, 40, 60, 80, &c. feet; and I should calculate the velocity with which each feeding sewer would bring its tributary waters into the main sewer, and calculate the continual acceleration, and so find at last the velocity at the outlet.

"It is true that the mean velocity is uniform, when the resistance arising from the friction of the channel is equal to the accelerating force which gives it motion. But if the accelerating force is *greater* than the resistance arising from friction, the mean velocity is not uniform, but *accelerated*, and we shall have a constantly increasing mean velocity from the head of the sewer to the outlet. I apprehend this is the case in practice. Not having Mr. Roe's gaugings, I cannot tell whether or no he has found an increasing mean velocity toward the outlet; but the facts would authorize one to say that it must be so.*

"Eytelwein's formula is inapplicable to this case: it refers only to an *open canal, where the mean velocity is uniform from the canal head to the outlet.*

"I would further suggest, that in estimating the quantity of water to be drained over a given surface, no allowance is made for evaporation; so that the quantity of water being less than that which Mr. Phillips calculates would drain away, it is not surprising that he found smaller drains than those he calculated, carrying off the surface waters of a larger district. The evaporation must be very considerable in summer; but I cannot refer to any tables

* The author has just received a note from Mr. Cowie, stating that he [illegible] Mr. Roe's gaugings of a case exactly in point, which confirms [illegible]

which [illegible]
course of the [illegible]
this manner.

"3. It is a [illegible]
a tube be [illegible]
itself so as to [illegible]

"If, [illegible]
be any [illegible]
streams [illegible]
All that [illegible]
the section [illegible]
fluid [illegible]
structure [illegible]
water to be [illegible]
outlet size [illegible]
convenient [illegible]
the tributary [illegible]
the drain [illegible]
final velocity [illegible]
the waters [illegible]
the [illegible]
system.

"If we [illegible]
itself above high-water [illegible]
of noxious gases [illegible]

"4. I am rather [illegible] siderations, that the [illegible] channel, or the velocity [illegible] the tube gets *line* [illegible] which remains stationary [illegible] only friction is the fluid rubbing against itself, the moving fluid against the stationary fluid. Mr. [illegible] has made experiments on this subject which show a greater difference than I had anticipated in favour of glazed tubes; and it probably may be the case, that in 'mere dribbles,' as the streams are in many cases, the glazed tubes would have an

stant accessions, moving with differe...
Cowie observes—

"I would calculate in the following ...
sewer five miles long, with a fall ...
other of 200 feet, *i.e.* a fall of ...
suppose it to receive additions
by sewers of given dimension.
these sewers will have fallen ...
feet; and I should calculate ... it is nece...
feeding sewer would bring ... of the column of ...
main sewer, and calculate ...
so find at last the velocity ... taken from Brees...

"It is true that the ...
resistance arising from ... the pump in inches, ...
to the accelerating force ... pounds avoirdupois of wat...
accelerating force is ... pipe; and one-tenth of t...
from friction, the me... ale gallons in a yard
lerated, and we shall
velocity from the he...
hend this is the case ... water will an engine have to lift,
gaugings, I cannot ... being 10 inches, and the length
increasing mean ...
would authori... weight of water in one yard's

"E...

refers ... equal 60 yards, we have
... 13 cwt. 2 qr. 8 lbs.

... water can be discharged
... and 270 feet perpendi-
... feet per minute?

... of the pipe.

... contained in the pipe.

... ale gallons per minute.

ON THE ARCH.

Coulomb was the first that considered the theory of the arch, with respect both to the sliding of voussoirs upon each other, and also to the rotation round the upper or lower edges of the joints.

The theory laid down by most English writers has only taken into account the sliding. Mr. Gwilt, one of our most learned English architects of the present day, has come forward in a very candid manner and acknowledged, that the theory of the arch, as taught by Emerson, Hutton, Atwood, and himself, is quite unsatisfactory, and has taken the theory as laid down by Rondelet, in his valuable work, "*Art de Batir.*" But Mr. Gwilt has assigned no sufficient reason why he has not availed himself of, nor even noticed, the researches of those excellent modern writers, Audoy, Petit, Persy, Poncelet, Garidel, Navier, Lamé, and Clapeyron, who, pursuing the theory of the arch first given by Coulomb, have raised a superstructure of science upon which every one, who is at all interested in the application of mathematics to arts of construction, must look with the highest degree of admiration.

The eminent engineers Lamé and Clapeyron investigate the equilibrium of the arch by considering it as a system of four levers, each lever being loaded with the weight of the respective part of the arch that corresponds with it: this method makes the calculation very laborious and difficult compared with the simple principle laid down by the illustrious Coulomb.*

* The author has, from the theory of Coulomb, deduced the same expression for the equilibrium, as is given by Gauthey on the principle of levers. (*See Theory, Practice, and Architecture of Bridges*, vol. i. pages 42, 43.)

For strict equilibrium,

The moment which tends to overturn the arch must be equal to the moment which tends to retain it.

If M be the weight of the whole semi-arch, including the pier; $A =$ the distance of the centre of gravity of the semi-arch and its pier, from the exterior edge F; $m =$ weight of the arch between the key and the point of rupture; $a = BL$, the horizontal distance between the point of rupture and the vertical passing through the centre of gravity of m; $h = aN$; $H = aR =$ the whole height; then $M \times A =$ moment of the semi-arch and its pier: this moment tends to move the semi-arch round F, as a fulcrum from left to right. $H \times$ maximum value of $\frac{ma}{h}$, is the moment that tends to turn the semi-arch and its pier round F, from right to left; hence, for strict equilibrium, if the maximum value of $\frac{ma}{h} = F'$, we have

$$F' \times H = M \times A, \text{ or } F' = \frac{MA}{H}.$$

When the values of m, a, and h are found corresponding to the maximum value of $\frac{ma}{h}$, then the ratio $\frac{h}{a}$ is the trigonometrical tangent of the angle which the tangent to the intrados makes with the horizon; which gives this beautiful property, viz. The point of rupture in an arch is that for which the tangent to the intrados at this point cuts the horizontal line passing through the summit of the key, at the same point as the vertical passing through the centre of gravity of the mass between the key and the point of rupture. This supposes that the joint of rupture is vertical, which is not strictly true, but the supposition favours the stability by making the horizontal thrust greater.

From this we have a practical method of finding the point of rupture of an arch.

Take any portion of the semi-arch, and draw a vertical

$\therefore$ the distance of the vertical through the centre of gravity from c

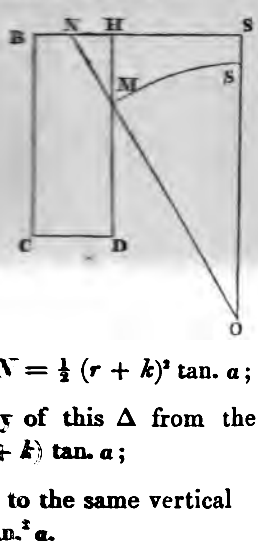
$$= r \sin. \phi - \tfrac{1}{3}\left(\frac{R^3 - r^3}{R^2 - r^2}\right) \cdot \left(\frac{1 - \cos. \phi}{\phi}\right) \; . \; . \; . \; (3.)$$

For the distance of the centre of gravity for a complete quadrant, we have $\phi = 90$; then equation (2) becomes

$$\tfrac{1}{3}\left(\frac{R^3 - r^3}{R^2 - r^2}\right) \cdot \frac{2}{\pi} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (4.)$$

Circular Arch with Horizontal Extrados.

Let $MD = H$;
$DC = e$;
$MH = 2h$;
$\angle NOS' = a$;
$OS = r$;
$SS' = k$;
$OS' = r + k$;
$S'H = r \sin. a$.



Then the area of the triangle $OS'N = \frac{1}{2}(r + k)^2 \tan. a$;

the distance of the centre of gravity of this Δ from the vertical line $OS = \frac{1}{3}(r + k) \tan. a$;

the moment of this Δ with respect to the same vertical line is $= \frac{1}{6}(r + k)^3 \tan.^2 a$.

Again, the area of the sector $OSM = \frac{1}{2} r^2 a$;

the distance of its centre of gravity from OS

$$= \frac{2\,r}{3\,a}(1 - \cos. a);$$

the moment of this sector with respect to OS'

$$= \frac{r^3}{3}(1 - \cos. a).$$

Now, the area of the space $NS'SM$

$$= \frac{1}{2}(r+k)^2 \tan. a - \frac{r^2 a}{2};$$

$\therefore$ distance of centre of gravity of $NS'SM$ from OS',

$$\frac{\text{moment}}{\text{area}} = \frac{(r+k)^3 \tan.^2 a - 2r^3(1-\cos. a)}{6\{\frac{1}{2}(r+k)^2 \tan. a - \frac{1}{2} r^2 a\}};$$

hence the distance of the centre of gravity of $S'SMN$ from BC,

$$e + r\sin. a - \frac{(r+k)^3 \tan.^2 a - 2r^3(1-\cos. a)}{3\{(r+k)^2 \tan. a - r^2 a\}}.$$

To find the moment of the part $BCDMN$ we must find the moment of the pier $BCDH$, and subtract the moment of the triangle MNH from it.

$$S'N = OS' \tan. a = (r+k)\tan. a;$$

$$HN = S'N - S'H = (r+k)\tan. a - r\sin. a;$$

area of the triangle $HNM = h\{(r+k)\tan. a - r\sin. a\}$;

the distance of the centre of gravity from HM

$$= \frac{1}{3} HN = \frac{1}{3}\{(r+k)\tan. a - r\sin. a\};$$

$\therefore e - \frac{1}{3}\{(r+k)\tan. a - r\sin. a\} =$ distance of the centre of gravity from BC;

$\therefore$ the moment of the triangle HNM

$$= h\{(r+k)\tan. a - r\sin. a\}.\{e - \tfrac{1}{3}[(r+k)\tan. a - r\sin. a]\}.$$

The moment of the pier $BCDH$

$$= BC \times CD \times \tfrac{1}{2} CD = (H+2h)\times e \times \tfrac{1}{2} e = \tfrac{1}{2} e^2 (H+2h);$$

$\therefore$ the moment of the part $BCDMN$

$$= \tfrac{1}{2} e^2 (H+2h) - eh\{(r+k)\tan. a - r\sin. a\}$$
$$+ \tfrac{1}{3} h\{(r+k)\tan. a - r\sin. a\}^2.$$

When the number of parts into which the semi-arch is divided is considerable, each of the parts may be regarded as a trapezoid; hence, the horizontal distance of the centre of gravity of the n^{th} trapezoid from B

$$= (n-1)\,a + \frac{(a_{n-1} + 2\,a_n)\,a}{3\,(a_{n-1} + a_n)};$$

$\therefore$ the moment of the n^{th} trapezoid

$$= \left\{ (n-1)\,a + \frac{(a_{n-1} + 2\,a_n)\,a}{3\,(a_{n-1} + a_n)} \right\} \times (a_{n-1} + a_n)\,\frac{a}{2}$$

$$= \{ (3\,n-1)\,a_n + (3\,n-2)\,a_{n-1} \}\,\frac{a^2}{6};$$

$\therefore$ the moment of the arch $BVad$

$$= \Sigma_{a_n}^{a_1} \{ (3\,n-1)\,a_n + 3(n-2)\,a_{n-1} \} \frac{a^2}{6}$$

$$= \{ a_0 + (3\,n-1)\,a_n + 6(a_1 + 2\,a_2 + 3\,a_3 + \ldots + \overline{n-1}\,a_{n-1}) \} \frac{a^2}{6}.$$

But the moment of the arch $BVad$

$$= \Sigma_{a_n}^{a_1} \{ a_n + a_{n-1} \} \frac{a}{2} \times BL$$

$$= \{ a_0 + a_n + 2\,(a_1 + a_2 + \ldots + a_{n-1}) \} \frac{a}{2} \times BL;$$

$$\therefore \{ a_0 + a_n + 2\,(a_1 + a_2 + \ldots + a_{n-1}) \} \frac{a}{2} \times BL$$

$$= \{ a_0 + (3n-1)\,a_n + 6(a_1 + 2a_2 + 3a_3 + \ldots + \overline{n-1}\,a_{n-1}) \} \frac{a^2}{6};$$

$$\therefore BL$$

$$= \frac{a\{ a_0 + (3\,n-1)\,a_n + 6(a_1 + 2a_2 + 3\,a_3 + \ldots + \overline{n-1}\,a_{n-1}) \}}{3\{ a_0 + a_n + 2(a_1 + a_2 + \ldots + a_{n-1}) \}}.$$

COR. The area of $BVad$

$$= \{ a_0 + a_n + 2\,(a_1 + a_2 + \ldots + a_{n-1}) \} \frac{a}{2}.$$

Hence th

$= \{ e + r \sin$

$\cdot \{ \frac{1}{2} (r +$

$+ \frac{1}{2} e^2 (H$

$+ \frac{1}{2} h \{ (r$

$= F \times D$

For the

the

For Semicircular Arches with Horizontal Extrados.

Value of the Ratio $\frac{R}{r}$	Angle of Rupture.	Ratio of the Thrust to the Square of the Radius of Intrados.
1.34	60°	0.14491
1.33	61	0.14467
1.32	61	0.14460
1.31	61	0.14390
1.30	61	0.14322
1.29	61	0.14264
1.28	62	0.14186
1.27	62	0.14101
1.26	62	0.13988
1.25	62	0.13872
1.24	62	0.13737
1.23	63	0.13593
1.22	63	0.13437
1.21	63	0.13263
1.20	63	0.13073
1.19	63	0.12870
1.18	63	0.12650
1.17	64	0.12415
1.16	64	0.12182
1.15	64	0.11895
1.14	64	0.11608
1.13	64	0.11303
1.12	64	0.10979
1.11	65	0.10641
1.10	65	0.10279
1.09	66	0.098992
1.08	66	0.094967
1.07	67	0.091189
1.06	68	0.086376
1.05	69	0.081755
1.04	70	0.076857
1.03	71	0.071853
1.02	73	0.066469
1.01	74	0.061324
1.00	75	0.055472

[illegible]	0.16683
[illegible]	0.16568
[illegible]	0.16448
[illegible]	0.16317
[illegible]	0.16167
[illegible]	0.16014
[illegible]	0.15845
[illegible]	0.15672
[illegible]	0.15482
[illegible]	0.15287
[illegible]	0.15096
[illegible]	0.14896
[illegible]	0.14678
[illegible]	0.14510
[illegible]	0.14330
[illegible]	0.14013
[illegible]	0.13691
[illegible]	0.13430
[illegible]	0.13157
61.13	0.12847
61.01	0.12516
[illegible]	0.12201
[illegible]	0.11887
[illegible]	[illegible]

MODULUS OF RUPTURE

[illegible]

For Semicircular Arches with PARALLEL *Extrados.*

Value of the Ratio $\frac{R}{r}$	Angle of Rupture.	Ratio of the Thrust to the Square of the Radius of Intrados.
1.44	64.30°	0.16683
1.43	64.00	0.16568
1.42	63.56	0.16448
1.41	63.52	0.16317
1.40	63.48	0.16167
1.39	63.43	0.16014
1.38	63.38	0.15845
1.37	63.32	0.15672
1.36	63.26	0.15482
1.35	63.19	0.15287
1.34	63.10	0.15096
1.33	63.10	0.14896
1·32	62.50	0.14678
1.31	62.33	0.14510
1.30	62.14	0.14330
1.29	62.09	0.14013
1.28	62.03	0.13691
1.27	61.47	0.13430
1.26	61.30	0.13157
1.25	61.15	0.12847
1.24	61.01	0.12516
1.23	60.40	0.12201
1.22	60.19	0.11887
1.21	60.00	0.11516
1.20	59.41	0.11140
1.19	59.10	0.10791
1.18	58.40	0.10417
1.17	58.09	0.10021
1.16	57.40	0.09593
1.15	57.01	0.09176
1.14	56.23	0.08729
1.13	55.45	0.08254
1.12	54.48	0.07789
1.11	54.10	0.07273
1.10	53.15	0.06754
1.09	52.14	0.06177
1.08	51.07	0.05649
1.07	49.48	0.05065
1.06	48.18	0.04455
1.05	46.32	0.03813
1.04	44.04	0.03139
1.03	41.04	0.02459
1.02	38.12	0.01691
1.01	32.36	0.00889

For Semicir
HORIZON

Value of the Ratio $\frac{R}{r}$	A Ru
1.34	
1.33	
1.32	
1.31	
1.30	
1.29	
1.28	
1.27	
1.26	
1.25	
1.24	
1.23	
1.22	
1.21	
1.20	
1.19	
1.18	
1.17	
1.16	
1.15	
1.14	
1.13	
1.12	
1.11	
1.10	
1.09	
1.08	
1.07	
1.06	
1.05	
1.04	
1.03	
1.02	
1.01	
1.00	

$$. . . 834 . (20.99)^2 - (16.4)^2 \quad (16.4 + x) - \frac{2}{3} \quad (20.99)^3 - (16.4)^3$$

$$134.79 (16.4 + x) - 1612.277$$

$$134.79\, x + 2210.556 - 1612.277$$

$$134.79\, x + 598.279.$$

... opposite $\dfrac{R}{r} = \dfrac{20.99}{16.44} = 1.28$, we hav

... the ratio of the horizontal thrust to the squa

... radius of the intrados;

$$\therefore F^* = .13691 \times (16.4)^2 = 36.8233;$$

... actual horizontal thrust is found by multiplying the value of

... by the square of the radius of intrados, in feet, and by th

... of a cubic foot of the material; thus, if the weight of a cubic foo

... material be 130 lbs., then

$$.13691 \times (16.4)^2 \times 130 = 4787 \text{ lbs.}$$

... the weight of the material would enter both side

... equation, therefore it is unnecessary to use it.

... the following rule for finding the horizontal thrust:—

... the ... the extrados by the radius of the intrados, and seek

... column of Table I. In the second column, in

... corresponding angle of rupture; in the

... of the horizontal thrust to the squar

... square of the radius of intrados, an

... the material: this product will giv

... ruction.

... angle of rupture found by the above rule

... rupture so found, to meet a tangen

... point of intersection draw the vertica

... centre of gravity of the part $a\,d\,B$ ...

... linear units as there are units of weigh

... ontal distance of the point so foun

... ontal thrust.

... ways give the horizontal thrust greate

... be a discrepancy between the con

... error is on the safe side, being i

and $36.8233 \times 27.55 = 1014.47 =$ [illegible] horizontal thrust:

$$\therefore 3.28\,x^2 + 134.79\,x - 59[illegible] = [illegible]$$

$$x^2 + 41.09\,x = 12[illegible]$$

$$\therefore x = 2.9 = \text{breadth of the pier.}$$

This agrees extremely nearly with [illegible] page 55, Theory of Bridges. (See Theory, Practice, and Architecture of Bridges, by Hann and Hosking.)

Given the height of the pier 2[illegible] [illegible] 10 feet; thickness at the crown [illegible] the pier 3.5 feet: will the pier [illegible] extrados being horizontal?

To find the centre of gravity of [illegible]

$a_0 = 11.5$:

$a_n = 1.5$:

$a_1 = 11.5 - \sqrt{18 \times 2} = 1[illegible]$

$a_2 = 11.5 - \sqrt{16 \times 4} = 1[illegible]$

$a_3 = 11.5 - \sqrt{14 \times 6} = 1[illegible]$

$a_4 = 11.5 - \sqrt{12 \times 8} = 1[illegible]$

$\therefore$ distance of the vertical through the centre of gravity of the semi-arch from the inner edge of the pier

$$= \frac{2}{3}\,\frac{\{11.5 + 14 \times 1.5 + 6[illegible]\}}{11.5 + 1.5 + 2\,[illegible]}$$

$$= \frac{2}{3} \cdot \frac{11.5 + 21 + 15[illegible]}{13 + 26.2} = \frac{192.[illegible]}{39.2} = [illegible]$$

$\therefore$ the distance of the vertical through the centre of gravity from the outer edge of the pier $= [illegible] = 6.[illegible]$

* The strict formula, page 24[illegible], is

$$\frac{\frac{1}{3}\,r\,(r + k)^2 - \frac{r^3}{2}}{r + k - \frac{\pi r}{4}} = \frac{57.[illegible]}{17.5} [illegible] \frac{[illegible]}{7.854} = 6.6[illegible] \text{ from } OA$$

$\therefore 10 - 6.64 = 3.36$ from HD, which is nearly the same as above.

∴ weight of the semi-arch $= 39.2 \times 120 = 4704$, by Cor. p. 243;

moment of the semi-arch $= 4704 \times 6.8 = 31987.2$;

moment of the pier $= 26 \times 3.5 \times 1.75 \times 120 = 19110$;

moment of the semi-arch and pier $= 51097.2$.

By the table,

$$\frac{F}{r^2} = .11895:$$

$$\therefore F = .11895 \times 100 \times 120 = 1427.4;$$

moment of the thrust $= 1427.4 \times 26 = 37112.4$.

The moment of the semi-arch and its pier being greater than the moment of the horizontal thrust, the arch will stand.

What must be the breadth of the pier so that it be upon the point of overturning?

$$\text{Moment pier} = 26 \times x \times \frac{x}{2} \times 120 = 1560\, x^2;$$

whole moment semi-arch and pier $= 31987.2 + 1560\, x^2$ must $=$ moment of thrust $= 37112.4$;

$$1560\, x^2 = 5125.2;$$
$$x^2 = 3.285;$$
$$x = 1.81.$$

... be greater than 1.44 for semicircular ... extrados, and greater than 1.34 for ... with horizontal extrados, this method ... cases, however, scarcely ever occur ... therefore rely on the methods here ... all practical purposes.

... see the theory fully investigated, ... stability, must consult the theory ... by Professor Moseley, in his ... *of Engineering and Architec-* ... *Mosking on Bridges.*

[illegible]

[illegible]

[illegible]

[illegible]

... covered with any body w...

... water were put into any ... temperature of the water ... began to boil, after wh... rise no further, as much hea... ...ar as is received from t... ...osed vessels, found that th... be raised to a very great ... ...rms us that he has raised w... ... melt tin.

...AM, ETC.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] increases. [illegible] = 7.40,

[illegible]

which [illegible] = [illegible] mentioned.

And [illegible]

in the [illegible]

the square of the radius of the roll, and extract the square root of the sum. Add this square root to the radius of the roll. Call the result the denominator of the first fraction.

Multiply the thickness of the rope by the square of the depth of the pit. Call the result the numerator of the second fraction.

Square the denominator of the first fraction before found, and multiply that square by 4 times 3.1416. Call the result the denominator of the second fraction.

Add the two fractions together.

Ex.—Whereabouts in a coal-shaft will the corves meet, if the radius of the roll be $3\frac{1}{2}$ feet, the thickness of the rope $\frac{1}{8}$ foot, and the depth of the pit 1020 feet.

By the rule,

$$3\tfrac{1}{2} \times 1020 = 3570;$$

$$\frac{1020 \times \frac{1}{8}}{3.1416} = 40.58441558';$$

$$3.5^2 = 12.25;$$

$$12.25 + 40.58441558' = 52.58441558';$$

$$\sqrt{52.58441558'} = 7.2686;$$

$$3.5 + 7.2686 = 10.7686;$$

hence $\frac{3570}{10.7686}$ is the first fraction.

Again,

$$\frac{1}{8} \times 1020^2 = 130050;$$

$$10.7686^2 \times 4 \times 3.1416 = 115.96274596 \times 12.5664 = 1457.234251;$$

hence $\frac{130050}{1457.234251}$ is the second fraction.

$$\frac{3570}{10.7686} + \frac{130050}{1457.234251} = 331.51941 + 89.2444 = 420.76381 \text{ feet} = 70.1273 \text{ fathoms.}$$

RULES FOR PARALLEL MOTION.

When the outer end of the radius rod is fixed, we must use the following rule :—

Rule.—Add the distance between the vertical line and the axis of the radius rod to twice the radius of the beam.

Divide the square of the radius of the beam by the above sum, and the quotient will be the length of the parallel bar.

If the radius rod be shorter than the parallel bar, divide the square of the radius of the beam by the difference instead of the sum.

The distance between the vertical line and the axis of the radius rod, added to or subtracted from the parallel bar, according as the radius rod is longer or shorter than the parallel bar, gives the length of the radius rod.

Or add the radius of the beam to the distance between the vertical line and axis of the radius rod; divide the square of this sum by twice the radius of the beam added to the above-named distance for the length of the radius rod.

Given the radius of the beam 12 feet, the distance between the vertical line and the axis of the radius rod = 4 feet.

By the rule $12 + 4 = 16$, then

$$\frac{16 \times 16}{2 \times 12 + 4} = \frac{256}{28} = 9.14 = \text{radius rod.}$$

When the length of the beam and the length of the parallel bar are given.

Subtract the length of the parallel bar from the length of the beam.

Divide the square of this difference by the length of the parallel bar, and the quotient will give the length of the radius rod.

Ex.—Suppose the length of the beam to be 10 feet, and the length of the parallel bar 4 feet, find the length of the radius rod.

Here $10-4=6=$ the difference mentioned in the rule.

$$\frac{6^2}{4}=\frac{36}{4}=9 \text{ feet, the length of the radius rod.}$$

A considerable saving of steam may be effected by working it expansively. Empirical formulæ have been given by Pambour, Pole, and Tate, which are good approximations; but, for the sake of simplicity, we shall here use Mariotte's law, which is sufficiently near for all practical purposes.

By Mariotte's law, the pressures are inversely as the spaces. The following rule is a good approximation.

Rule.—Divide that part of the stroke through which the expansion takes place, into any even number of equal parts, and calculate the pressure per square inch upon the piston at each division of the stroke; take the sum of the extreme pressures in pounds per square inch, four times the sum of the even pressures, and twice the sum of the odd pressures; multiply the sum of all these by one third of the common distance between the positions of the piston, and the result will be the work done upon each square inch of the piston after expansion begins. The work done before the expansion begins is evidently equal to the pressure per square inch multiplied by the number of feet described before expansion. The whole work done during a single stroke is equal to the sum of the work done before and after expansion.

Ex.—The pressure of steam upon the piston is 40 lbs. per square inch, the resistance arising from imperfect condensation 3 lbs. per square inch, the length of the stroke 12 feet, and the steam is cut off at one $\frac{1}{6}$ of the stroke; find the number of units of work done upon each square inch of the piston, and the number of units of work gained by working expansively. Also find the load per square inch, and the position of the piston when the velocity is greatest.

Divide the remaining part of the stroke, viz. 10 feet, into 10 equal parts.

$$3 : 2 :: 40 : P_1$$

$$P_1 = \frac{2 \times 40}{3} = 26.666.$$

In the same way we have,

$$P_2 = \frac{2 \times 40}{4} = 20. \qquad P_6 = \frac{2 \times 40}{8} = 10.$$

$$P_3 = \frac{2 \times 40}{5} = 16. \qquad P_7 = \frac{2 \times 40}{9} = 8.888.$$

$$P_4 = \frac{2 \times 40}{6} = 13.333. \qquad P_8 = \frac{2 \times 40}{10} = 8.$$

$$P_5 = \frac{2 \times 40}{7} = 11.428. \qquad P_9 = \frac{2 \times 40}{11} = 7.273.$$

$$P_{10} = \frac{2 \times 40}{12} = 6.666.$$

$40 + 6.666 = 46.666$ = sum of extreme pressures,

$$\begin{array}{r l}
26.666 & \\
16.000 & \\
11.428 & \\
8.888 & \\
\underline{7.273} & \\
70.255 & = \text{sum of even pressures.} \\
\underline{4} & \\
281.020 & = \text{four times the sum of even pressures.} \\
20.000 & \\
13.333 & \\
10.000 & \\
\underline{8.000} & \\
51.333 & = \text{sum of the odd pressures.} \\
\underline{2} & \\
102.666 & = \text{twice the sum of odd pressures.} \\
281.020 & \\
\underline{46.666} & \\
3)\underline{430.352} & \\
143.451 & = \text{work done by expansion.} \\
\underline{80} & = \text{work done before expansion.} \\
\underline{223.451} & = \text{whole work done upon each square inch.}
\end{array}$$

radius by the cube of the number of revolutions per minute.

Divide the former product by the latter, and the quotient will be the weight in tons.

To find the Mean Radius of the Wheel.

Rule 2.—Multiply the number of horse-power by *n*,* divide the product by the area of the section of the rim, and extract the cube root of the quotient.

Divide 12.17 by the number of revolutions per minute, and multiply the quotient by the cube root before obtained.

The product will be the mean radius required.

To find the Area of the Section of the Rim.

Rule 3.—Multiply 1802.9 by the number of horse-power, and that product by (*n*). Multiply the cube of the mean radius by the cube of the number of revolutions per minute.

Divide the former product by the latter, and the quotient will be the area of the section.

---

* Morin, at page 191 of the Aide Mémoire de Mécanique Pratique, says that "(*n*) should be taken = 20 or 25 for engines which are not required to work with very great velocity, such as flour-mills, saw-mills, &c. For engines working spinning or weaving machines, (*n*) should be taken 35 or 40. Where the spinning is to be done with very great regularity, (*n*) should be 50 or 60."

STRENGTH AND STRESS OF MATERIALS.

A knowledge of the strength and stress of materials is of so much importance to the practical mechanic, that it demands his most serious attention; and he will find the works of Barlow and Tredgold* to be treasures of inestimable value, being fraught with every kind of useful information relating to these subjects.

Mr. Barlow shows that there are four distinct strains to which every hard body may be exposed, and which may be stated as follows:—

1. A body may be pulled or torn asunder by a stretching force, applied in the direction of its fibres; as in the case of ropes, stretchers, king-posts, tie-beams, &c.

2. It may be broken across by a transverse strain, or by a force acting either perpendicularly or obliquely to its length; as in the case of levers, joists, &c.

3. It may be crushed by a force acting in the direction of its length; as in the case of pillars, posts, and truss-beams.

4. It may be twisted or wrenched by a force acting in a circular direction; as in the case of the axle of a wheel, the nail of a press, &c.

ON THE COHESIVE STRENGTH OF MATERIALS.

The force of cohesion may be defined to be that force by which the fibres or particles of a body resist separation, and is therefore proportional to the number of fibres in the body, or to the area of its section.

---

* Barlow on the Strength and Stress of Timber, and Tredgold on the Strength of Cast Iron.

Mr. Emerson gives the load that may be safely borne by a square-inch rod of each of the following:—

	Pounds Avoirdupois.
Iron rod, an inch square will bear	76,400
Brass	35,600
Hempen rope	19,600
Ivory	15,700
Oak, Box, Yew, Plum-tree	7,850
Elm, Ash, Beech	6,070
Walnut, Plum	5,360
Red Fir, Holly, Elder, Plane, Crab	5,000
Cherry, Hazel	4,760
Alder, Asp, Birch, Willow	4,290
Lead	430
Freestone	914

He also gives the following practical rule, viz. That a cylinder, the diameter of which is d inches, loaded to one-fourth of its absolute strength, will carry as follows:—

	cwt.
Iron	$135 \times d^2$
Good Rope	$22 \times d^2$
Oak	$14 \times d^2$
Fir	$9 \times d^2$

Also he says that a cylindric rod of good clean fir, of an inch circumference, drawn in length, will bear at its extremity 400 lbs.; and a spear of fir, 2 inches diameter, will bear about 7 tons, but not more.

A rod of good iron, of an inch circumference, will bear near 3 tons weight.

A good hempen rope, of an inch circumference, will bear 1000 lbs. being at its extremity.

The following interesting experiment was made at the Patent Iron Cable Manufactory of Capt. S. Brown (see Barlow's Essay, 2d edition, p. 257):—

A bolt of Welsh iron, 12 feet 6 inches long, and 2 inches

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	.113
[illegible]	[illegible]
[illegible]	.111
[illegible]	[illegible]
[illegible]	[illegible]
	[illegible]
	[illegible]

[illegible] the ultimate [illegible] strength of [illegible] piece of timber, fixed at one end and loaded at the other. [illegible] Multiply the value given in the table of [illegible] strength and square of the depth, both in inches, and divide that product by the length, also in inches, and the quotient will be the weight in pounds.

[illegible] — What weight will be required to break a piece of [illegible] forest [illegible] the breadth being [illegible] inches [illegible] and length [illegible] feet?

Then $\frac{1672 \times 2 \times 6^2}{144} = 836$ lbs.

Or, if the dimensions of a beam be required, so as to support a given weight at its end, then—

Multiply the weight in pounds by the length in inches; and this product, divided by the tabular value, will give the product of the breadth and square of the depth.

Ex. 1.—Required the dimensions of a beam of larch, 10 feet long, so as to be capable of supporting a weight of 1,000 lbs. at one end, the other end being fixed in a wall.

The tabular value for larch is 1,127.

Therefore $\frac{120 \times 1000}{1127} = 106.5$ nearly, = the breadth and square of the depth.

Let the breadth be 2 inches, then $\frac{106.5}{2} = 53.25$, the square of the depth; and $\sqrt{53.25} = 7.3$ inches the depth.

Ex. 2.—A square balk of ash projects 4 feet 6 inches from a solid wall in which it is fixed; what must be the side of its square, so that the balk may be able to support 1,013 lbs.?

The tabular value for ash is 2,026.

$\frac{54 \times 1013}{2026} = \frac{54702}{2026} = 27$, the cube root of which is 3 inches, the side of the square required.

Case 2.

To compute the ultimate transverse strength of any rectangular beam, when supported at both ends, and loaded in the centre.

Rule.—Multiply the value given in the table of data by four times the breadth and square of the depth in inches, and divide that product by the length, also in inches, for the weight.

Ex. 1.—What weight will be necessary to break a beam of Canadian oak, the length being 10 feet, the breadth 6

Also, a cylinder of two inches in diam[illegible]
times as much as a cylinder one i[illegible]
The following table of data is [illegible]
Barlow's Essay :—

Teak	[illegible]
English Oak	[illegible]
Canadian do.	[illegible]
Dantzic do.	[illegible]
Adriatic do.	[illegible]
Ash	[illegible]
Beech	[illegible]
Elm	[illegible]
Pitch Pine	[illegible]
Red Pine	[illegible]
New England F[illegible]	[illegible]
Riga Fir	[illegible]
Mar Forest Fir	[illegible]
Larch	[illegible]

[illegible] lbs.
[illegible] break
[illegible] 6 inches
[illegible] is 1,262.
$= \frac{1262 \times [illegible]}{24}$

[illegible] required, so as
[illegible] must be used.
[illegible] pounds by the
[illegible] by the tab[illegible]
[illegible] the breadth [illegible]
[illegible] being know[illegible]
[illegible] known, w[illegible]

To find the [illegible]
beam [illegible] a beam
Rule [illegible] to bea[illegible]
by the br[illegible]
and divide [illegible] is 1,157.
the qu[illegible] the squ[illegible]
Ex. [illegible]
Mar F[illegible]
and [illegible] $\times 4 = 20$
In [illegible]
1,262 [illegible]
Th[illegible] [illegible] ed unifor[illegible]
E. [illegible] the midd[illegible]
Ex[illegible]
inc[illegible] [illegible] formly th[illegible]

... the depth

... when b = ...

$= 1.4687$

ON TRANSVERSE STRENGTH

CASE ...

... To find the breadth ...
... to bear a given weight ...
... ply the length of the ...
... supported in pounds, ...
... the square of the depth ...
... the breadth in inches.

Rule 2.—To find the de...
... to bear a given weight ...
Multiply the length of the ...
to be supported in pounds ...
times the breadth ...
quotient will be the depth ...

When no particular ...
the nature of the ...
it will be found ...
portion; as, for example ...
of the depth, ...
the rule becomes—

Rule 3.— ...
weight ...

inches, and the depth 10 inches; being ... required, and end, and loaded in the middle? ... depth.

For Canadian oak the tabular value is ... for inclined as for

$$\frac{1766 \times 4 \times 6 \times 10^2}{120} = \frac{4238400}{120} = \text{... distance between ... of bearing.}$$

Ex. 2.—What weight will it require ...
Mar Forest fir, which is 5 inches broad ...
20 feet between the supports? ... the middle between the
The tabular value for Mar Forest ...

By the rule, $\frac{1262 \times 4 \times 5 \times ...}{240}$ ... the weight from one ... of the weight from the
3786 lbs. ... this product divided by

If the dimensions of a beam ... supports, will give the effec-
a given weight, the following ... feet; which being used in-

Rule.—Multiply the weight ... of the foregoing rules, the
inches; and this product ... found by them.
will give the product of four ...
of the depth: therefore, the ... CASE 3.
find the depth; or, the ... equally distributed over the length
the breadth. ... being supported at both ends.

Ex.—What must be the ... rules apply as in Case 1; only, in-
oak, 20 feet long between ... 550, divide by 1700, which is the
of 2 tons in the middle of ...

The tabular value for ... CASE 4.

$$\frac{4480 \times 240}{1457} = 670 ...$$

... is fixed at one end, and the load
depth multiplied by ... other; also, when a beam is
Let the breadth be ... motion.
the breadth. ... take *B C* for the length; or
... the middle take *B C* or *B C'*
... rules given in Case 1; only,
... divide by 212.
... take 238 for a divisor.
... distributed over the length
... divisor, instead of 850, in the

* If the beam be ... out its length, the result ...
If the beam be fixed ... must be increased by ...
If the beam be ... length, the result must ...

[illegible]

[illegible] 20 tons.

[illegible] = [illegible]

Then, by Rule [illegible] [illegible] = [illegible]

[illegible]—What is the [illegible]

[illegible] point?

[illegible] = [illegible]

By Rule [illegible], $\frac{1120 \times [illegible]}{850 \times 4}$ = [illegible]

[illegible] is $5\frac{1}{4}$ inches, the [illegible]

Ex. 3.—What are the [illegible] which is capable of [illegible] middle point, the depth [illegible]

In this example we [illegible]

Here [illegible] = 4. [illegible] = [illegible]

Then 40 × 4 = 160 = [illegible]

$\frac{11200 \times 160}{850}$ = [illegible] the depth in inches, [illegible] = [illegible] inches.

Ex. 4.—If the depth of [illegible] what will these dimensions be when the [illegible] the beam is 20 feet, and a weight of [illegible] tons is supported at 8 feet from one end?

Here we must take Rules [illegible] and 4.

$\frac{12 \times [illegible] \times 4}{12}$ = 32 = the effective leverage.

And since the depth is equal to three times the breadth, $n = 3$; and, by Rule 3, 32 × 3 = 96.

$\frac{14800 \times 96}{850}$ = [illegible], the cube root of which is 17. [illegible] depth.

Ex. 2.—What must be the diameter of a cast iron shaft to resist a pressure of 2000 lbs. at 2 feet from the end, the whole length of the shaft being 7 feet?

Since the load is applied at 2 feet from one end of the shaft, it must be 5 feet from the other.

$\frac{2 \times 5 \times 4 \times 2000}{500 \times 7} = 22.8$, the cube root of which is $2\frac{5}{8}$ inches, the diameter required.

ON GUDGEONS.

In gudgeons, one-fifth of the diameter is usually allowed for wear; and, on this principle, Mr. Tredgold gives the following rule:—

Multiply the stress in pounds by the length in inches; and the cube root of the product, divided by 9, is the diameter of the gudgeon in inches.

Ex.—If the stress on a gudgeon be 10 tons, and its length 7 inches, what is the diameter?

$$10 \text{ tons} = 22400 \text{ lbs.}$$

$7 \times 22400 = 156800$, the cube root of which is 54 nearly; and $\frac{54}{9} = 6$ inches, the diameter required.

ON THE FORMS OF BEAMS.

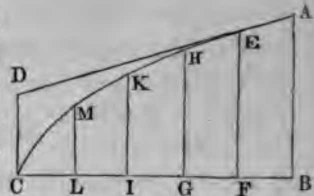
In the construction of beams, it is necessary that their form should be such that they will be equally strong throughout; or, in other words, that they will offer an equal resistance to fracture in all their parts, and will therefore be equally liable to break at one part of their length as at another.

If a beam be fixed at one end and loaded at the other, and the breadth uniform throughout its length, then, that the beam may be equally strong throughout, its form must be that of a parabola.

This form is generally used in the beams of steam-engines; and, in double-acting steam-engines, the beam is strained sometimes from one side, and sometimes from the other; therefore both the sides should be of the same form.

The crank, as used in the steam-engine, should be of the same form.

Dr. Young and Mr. Tredgold have considered that it will answer better, in practice, to have some straight-lined figure to include the parabolic form; and the form which they propose is to draw a tangent to the point A of the parabola ACB.



But as few practical men understand how to draw a tangent to a parabola, or even a parabola itself, we will here show how they may do both.

We will, in the first place, show how to draw a parabola.

Let CB represent the length of the beam, and AB the semi-ordinate, or half the base; then, by the property of the parabola, the squares of all ordinates to the same diameter are to one another as their respective abscissas.

Now, if we take $CB = 4$ feet, and $AB = 1$ foot, we may proceed to apply this property to determine the length of the semi-ordinates corresponding to every foot in the length of the beam, as follow:—

$$CB : CF :: AB^2 : EF^2;$$
that is, $48 : 36 :: 12^2 : 108 = EF^2$;
the square root of which is 10.4 nearly $= EF$.

And $CB : CG :: AB^2 : GH^2$;
$$48 : 24 :: 12^2 : 72 = GH^2;$$
the square root of which is 8.5 nearly $= GH$.
$$CB : CI :: AB^2 : IK^2;$$
$$48 : 12 :: 12^2 : 36 = IK^2;$$
the square root of which is 6 inches $= IK$.

Now, if we take $CL = 6$ inches,
then $CB : CL :: AB^2 : LM^2$;
$$48 : 6 :: 12^2 : 18 = LM^2;$$

the square root of which is 4.24, which is very near $4\frac{1}{4}$ inches $= LM$.

Now, if any flexible rod be bent so as just to touch the tops A, E, H, K, M, of the ordinates, and the vertex C, then the form of this rod is a parabola.

To draw a tangent to any point A of a parabola:—

From the vertex C of the parabola draw CD perpendicular to CB, and make it equal to $\frac{1}{2}AB$; then join A, D, and the right line AD will be a tangent to the parabola at the point A; that is, it touches the parabola at that point.

In the same manner, we may draw a tangent to the parabola at any other point, by erecting a perpendicular at the vertex equal to half the semi-ordinate at that point.

When a beam is regularly diminished towards the points that are least strained, so that all the sections are similar figures, whether it be supported at each end and loaded in the middle, or supported in the middle and loaded at each end, the outline should be a cubic parabola.*

When a beam is supported at both ends, and is of the same breadth throughout, then, if the load be uniformly distributed throughout the length of the beam, the line bounding the compressed side should be a semi-ellipse.

The same form should be made use of for the rails of a waggon-way, where they have to resist the pressure of a load rolling over them.†

BEAMS OF PUMPING ENGINES.

By page 277, if a beam be fixed at one end and the load applied at the other, or, which is the same thing, if a beam be supported on a centre of motion, then the figure of equal strength is a parabola, the breadth of the beam being the same throughout.

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* Gregory's *Mechanics*, Vol. I. Art. 181; or Tredgold on *Cast Iron*, page 48.

† Tredgold, page 49.

Ex.—If the force acting upon a crank be 6000 lbs. and its length be 3 feet, what are its breadth and depth, so that the deflection may not exceed one-tenth of an inch.

$\frac{6000 \times 3^3}{2662 \times .1} = 610$ nearly, $=$ breadth multiplied by the cube of the depth.

If the breadth be made 3 inches, the depth should be nearly 6 inches, for the cube of $6 \times 3 = 648$.

If the depth at the end where the force acts be half the depth at the axis, divide by 1628 instead of 2662.

$\frac{6000 \times 3^3}{1628 \times .1} = 1000 =$ breadth multiplied by the cube of the depth.

If we make the breadth 4 inches, then $\frac{1000}{4} = 250$, the cube root of which is 6.3 inches, the depth required.

WHEELS.

Multiply the weight or power, in pounds, acting at the end of the arm, by the cube of its length in feet; and this product, divided by 2662 times the number of arms multiplied by the deflection, will give the product of the breadth and cube of the depth.

When the depth at the rim is half that at the axis, divide by 1628 instead of 2662.

Ex.—If the force which acts at the circumference of a spur wheel be 1600 lbs. the radius of the wheel 6 feet, and the number of arms 8, and let the deflection not exceed $\frac{1}{10}$ of an inch; required the breadth and depth.

$\frac{1600 \times 6^3}{2662 \times 8 \times .1} = 162\frac{1}{4} =$ breadth and cube of the depth.

If the breadth be made 2.5 inches, then $\frac{162.25}{2.5} = 64.9$, the cube root of which is 4.018, the depth.

Note.—These rules are formed from the formulæ given in Tredgold's Essay on Cast Iron.

OF TORSION.

The resistance which a shaft or any other body offers to a force applied to twist it round, is called the resistance to torsion.

The strength of cylinders to resist torsion or twisting, is generally estimated to be proportional to the cubes of the diameters.

In the Additions to Buchanan on Mill-work, by Mr. Tredgold, the following formula is given for cylindrical shafts of cast iron to resist torsion:—

$$\sqrt[3]{\frac{240\,h}{n}} = d;$$

where h denotes the number of horses' power, n the number of revolutions of the shaft per minute, and d its diameter in inches.

This expressed in words—Multiply the number of horses' power by 240, and divide this product by the number of revolutions of the shaft per minute; then the cube root of the quotient will give the diameter of the shaft in inches.

[illegible] shaft makes 3[illegible] revolutions per minute, [illegible] 2[illegible] horses, required the [illegible] to resist the torsion.

[illegible] = [illegible] the cube root of which [illegible] diameter required.

[illegible] sustain lateral pressure and [illegible]

[illegible] = [illegible] diameter in inches.

[illegible] length in feet between the bearings. H the [illegible] power which are equal to the first mover, [illegible] revolutions to be made by the shaft per [illegible] stress it owns. See Tredgold's [illegible] Mill-work, page 382.

[illegible] shaft of cast iron make 40 revolu-[illegible] power being equal to 7 horses,

the length of the shaft 30 feet, and the [illegible] required the diameter of the shaft.

By the formula,

$$\sqrt[3]{\left(\frac{[illegible]}{[illegible]} - \frac{[illegible]}{[illegible]}\right)} = [illegible] - [illegible] = [illegible] = [illegible]$$

inches nearly.

It is necessary, before taking leave of this subject, to call the attention of the practical mechanic to the distinction between stiffness and strength.

Stiffness may be defined to be that property which resists flexure or bending; and strength has already been shown to be that which resists fracture. Therefore, the limit of stiffness is flexure, and the limit of strength is fracture.

Beams of equal lengths have their lateral stiffness to bear a load at any point in the length, as the breadth and cube of the depth.* But their lateral or transverse strengths are as the breadth and square of the depth.

Thus, if a beam be two feet square, it will sustain sixteen times as much weight, without bending, as a beam one foot square.

But a beam two feet square will only be eight times stronger than a beam one foot square, the beams being of the same length.

Beams of different lengths have their stiffness (to bear a load at any point of their lengths) directly as the breadth and the cube of the depth, and inversely as the cube of the length; and have their strength directly as the breadth and the square of the depth, and inversely as the length.†

It may be necessary to remark here, that the rules for the strength, &c. of timber, are taken from Barlow's Essay, and those on cast iron from Tredgold's Essay.

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* Dr. Young's Natural Philosophy, vol. i. p. 139.

† Ibid. vol. ii. Arts. 333, 335.

the sum of these gives the transverse strength of the beam

$$= \frac{Pby^2}{3} + \frac{Qb\,(a-y)^2}{3}$$

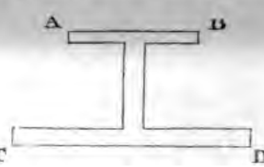
$$= \frac{Pby}{3}\,(y + a - y), \text{ since } Py = Q\,(a-y)$$

$$= \frac{Pbya}{3}.$$

Now, ba is the area of the section; hence Pba is the direct force of the beam.

The transverse strength of a beam is found by multiplying the direct strength by the depth of the neutral line, and dividing by 3.

Professor Hodgkinson tried a great many experiments to find the strongest form of section for a cast-iron beam, and the result shows that the lower flanch must have about six times the material of the upper, as in the figure.



For columns of the strongest form, he finds that the strength of a column of cast-iron, containing a given weight of material, whether it be hollow or solid, is much greater when it is cast in the form of a double cone—the greatest thickness being in the middle, and tapering towards both ends.

If a column be rounded at both ends, it will only support one-third of the weight of a similar column whose extremities are both flat; and when one end is rounded, and the other flat, it requires a breaking-weight of two-thirds. The strengths of these three columns are therefore as the numbers 1, 2, 3—a remarkable result of great practical importance; for, by an injudicious form of the columns at the ends, two-thirds of the metal may be thrown away.

The same eminent writer and experimentalist, speaking of those theories which suppose that bodies, when not

overstrained, are perfectly elastic, and resist extension and compression with equal energy, observes: "But theories deduced from these suppositions, however elegant and nearly correct for small displacements of the fibres or particles, give the breaking strength of cast iron in some cases not half what it has been shown to bear by experiment. A square bar, instead of having its neutral line in the centre—one-half being extended, and the other compressed, according to the supposition above—requires to be considered as totally incompressible; the neutral line being close to the side, or even beyond it. This defect in the received theories has been shown to arise from the neglect by some writers of an element which appears to be always conjoined with elasticity, diminishing its power: this element—ductility—producing defective elasticity."

In a rectangular or circular body, considered as perfectly elastic, the neutral line would be in the middle; but, as cast iron resists compression with much more energy than extension, the neutral line of that material, in a bent body, will be nearer to the compressed side—and to such a degree, that the exploded hypothesis of Galileo, where the neutral line was considered to be at the edge, would give results more in accordance with experiment, than the rules laid down by authors on the supposition of perfect elasticity.

Mr. Hodgkinson gives the following rule for calculating the weight necessary to break a beam of strongest form:—

Multiply the area of the section of the lower flanch, in inches, by the depth of the beam, and divide the product by the distance between the two points of support. This quotient, multiplied by 536 when the beams are cast erect, and by 514 when they are cast horizontally, will give the breaking weight in cwts.

APPENDIX

To estimate the Work done, &c.

Let ABC be a line along which …
B is between A and C.

AB …

AC …

then the force varies as the …
it be supposed increas…

Let us take the … &c.,
the force to be cons… expres…
through it.

∴ the common …
for BC.

Let P be the val… each for…
Then P is variabl… number of …
Let W be the wh… …ed,
moves from A to …
A to C.

Then $W = W$ … ΣP …;

and $P(BC) =$ … ΣP … &c.

The [illegible]
coro[illegible]
city = [illegible]

[illegible]

or, if [illegible]
by its [illegible]

This is the [illegible]

To [illegible]
done.

We shall [illegible]
equation [illegible]

Let [illegible]

Then [illegible]

but, [illegible]

[illegible]

[illegible]

particles.

[illegible]

[illegible] consider the [illegible] of the equation.

contact: this product will give the friction $P.f.$ Multiply this friction by the space passed over in one revolution, or by the circumference $2\pi r = 6.28 r$ of the axis: the product $6.28 P.f.r$ is the work expended in each revolution.

This product, multiplied by the number of revolutions per second, will give the work expended on friction per second; *i. e.*

$$6.28 \cdot P.f.r.n = \text{work of friction per second.}$$

Given the radius axis = 6 inches; the weight of the shaft, and other parts pressing on it = 1000 lbs.; the shaft makes 10 revolutions.

f, the co-efficient of friction in the tables, is .07;

$$Pf = .07 \times 1000 = 10 \text{ lbs.};$$

the space passed over by the circumference of the axis is

$$\frac{6.28 \times .5 \times 10}{60''} = .523 \text{ feet};$$

hence the quantity of work expended on friction is

$$10 \text{ lbs.} \times .523 = 5.23.$$

Friction of a Pivot.

Rule.—Multiply the pressure P by the ratio f of the friction to the pressure: this product will give the friction.

Multiply this product by two-thirds of the exterior circumference of the base of the pivot, or by $4.19 r$.

The product, $4.19 f P r$, will be the work expended in each revolution by the friction of the pivot.

To have the work expended per second, multiply the above product by the number of revolutions per second.

$4.19 n f P r$ is the work expended on friction, n being the number of revolutions per second.

volume of Bx in cubic feet $= 2ax$, and its weight $= 2\mu ax$;

$$\therefore CI = 2\mu ax + P\cos. a;$$

$$\therefore \frac{Qm}{Cm} = \frac{P\sin. a}{2\mu ax + P\cos. a};$$

$Qm = Qy - my = y - a$, and $Cm = Am - Ac$ if $b =$ the distance AC from the summit of pier, which distance the direction of P intersects the axis;

$$\therefore \frac{y-a}{x-b} = \frac{P\sin a}{2\mu ax + P.\cos. a};$$

$$\therefore y - a = \frac{P(x-b)\sin. a}{2\mu ax + P.\cos. a},$$

$$\text{or } y = a + \frac{P(x-b)\sin. a}{2\mu ax + P.\cos. a},$$

ich is the equation to the line of resistance in a pier.

If the resultant R intersect the plane of the section xy, a point without the surface of the pier, it is evident that pier will upset, for the portion Bx will be pressed on subjacent part of the pier by a single force R, whose ection is without the base, and which therefore cannot held in equilibrium by the resistance of the base, since t resistance cannot be exerted in a direction opposed to the pier will therefore be overthrown whenever the e of resistance intersects the surface; and it will be rthrown at that point where this line first intersects the face. Now the line of resistance intersects the posterior face when $y = 2a$. We can determine the depth x at ich this intersection takes place; therefore, by making $= 2a$, or $y - 2a = 0$, the distance x thus determined he greatest height to which the pier can be built with- t upsetting.

Suppose h to represent this height;

$$\therefore 2a = a + \frac{P(h-b)\sin. a}{2\mu ah + P.\cos. a};$$

$$2\mu a^2 h + P.a\cos. a = P.h\sin. a - P.b\sin a;$$

$$\therefore\ h(2\mu a^2 - P\sin.\alpha) = -P(a\cos.\alpha + b\sin.\alpha);$$

$$\therefore\ h = \frac{P.(a\cos.\alpha + b\sin.\alpha)}{P\sin.\alpha - 2\mu a^2}$$

General equation to the line of resistance is

$$y - a = \frac{P(x-b)\sin.\alpha}{2\mu a x + P.\cos.\alpha}.$$

If the force P be applied at the summit of the pier, then $b = 0$, and we have

$$y - a = \frac{Px\sin.\alpha}{2\mu a x + P.\cos.\alpha}.$$

If $\alpha = \frac{\pi}{2}$, or the force P be applied horizontally, then $\cos.\alpha = 0$, and $\sin.\alpha = 1$;

$$\therefore\ y - a = \frac{Px}{2\mu a x} = \frac{P}{2\mu a};$$

$$\therefore\ y = a + \frac{P}{2\mu a};$$

from which it follows that the line of resistance in this particular case resolves itself into a vertical straight line, situated at a horizontal distance beyond the axis of the pier, which is represented by $\frac{P}{2\mu a}$.

Let us now suppose that instead of the line of resistance intersecting the extrados of the pier at the height h, so that that height may represent the greatest height to which the pier may be built; it may intersect it (the pier) at a distance of c feet within the mass of the pier. Let the height of the pier under these circumttances $= H$.

Then when $x = H$, $y = 2a - c$;

$$\therefore\ \text{substituting } 2a - c - a = \frac{PH\sin.\alpha}{2\mu a H + P\cos.\alpha},$$

$$\text{or, } 2\mu a H(a-c) + P\cos.\alpha\,(a-c) = P.H.\sin.\alpha;$$

$$\therefore\ H = P\cos.\alpha . \frac{c-a}{2\mu a(a-c) - P.\sin.\alpha}.$$

Friction of Cylinders.

The proportion which the friction of a large cylinder bears to the friction of any number, the sum of the areas of which is equal to the area of the large cylinder, may be shown as follows:—

Let d = the diameter of one of the small cylinders;

n = the number of them.

Then $d^2 n \frac{\pi}{4}$* = area of large cylinder;

$\therefore d\sqrt{n}$ = diameter of large cylinder.

But the friction is proportional to the circumference of the cylinder; therefore the friction of the small cylinders may be represented by πdn, and the friction of the large cylinder by $\pi d\sqrt{n}$. Hence

the friction of the large cylinder : the friction of all the small cylinders :: $\pi d\sqrt{n} : \pi dn$;

$:: \sqrt{n} : n$;

which, in the case of 4 small cylinders, becomes 1 : 2; that is, the friction of 4 cylinders is double the friction of one cylinder, the area of which is equal to the sum of the areas of all the four.

Formulæ for the Centre of Gravity of a Body symmetrical with respect to the axis of x.

Let x be the distance of any particle da from the axis of y; then $x\,da$ = its moment with respect to that axis, and $\int x\,da$ = the sum of the moments of all the particles. Let X be the distance of the centre of gravity from the same axis; then—

---

* π stands for the circular measure 3.14159.

MISCELLANEOUS EXAMPLES.

• A HEAVY beam rests upon a peg, with one end against nooth vertical wall. Find the position of equilibrium.

2. A heavy beam lies partly in a smooth hemispherical vl, and partly over one edge. Find the position of uilibrium.

3. The height of an embankment sustaining water = feet; breadth = 1.5 feet; weight of a cubic foot of the terial = 130 lbs.; it is supported by a stay 5 feet long, lower end being placed 3 feet from the bottom. Find thrust on the stay, when the embankment is upon the nt of overturning.—6768.

Here the moment of the thrust added to the moment the embankment will be equal to the moment of the essure of the water.

4. Find the *vis viva* of a circular disc thrown from the nd with a velocity v, and angular velocity a.

$$v = \frac{a}{2}.$$

5. A sphere is sustained upon an inclined plane by the essure of a beam about the lowest point of the inclined ane. Given the position of the beam, required that of e plane.

6. A uniform beam rests with its extremities against a rizontal and smooth vertical wall. Find the pressures the extremities.

7. There is a river wall 5 feet thick, and 20 feet high; e water stands at 8 feet from the top. Required the odulus and pressure upon each foot of length, the wall ing granite.—Modulus 1.15 feet; press. 7031.25.

[illegible]

[illegible] with the [illegible]
[illegible] the effective [illegible]

[illegible] maximum [illegible] $= $ [illegible]

[illegible]

[illegible]

ascend an incline of 1 in 100, taking friction at 8 lbs. per ton?—*Ans.* .74 miles.

17. Two beams, connected together at a given angle, turn about a horizontal axis at their point of meeting; find the position of equilibrium which they will take by the action of their own weight.

18. Two balls, each weighing 50 lbs., are placed at the extremities of a horizontal arm, which gives motion to a screw driving a punch, as in the common stamping machine. The velocity given to the balls is 10 feet per second; it is required to find the mean resistance opposed to the punch, when it is just driven through an iron plate $\frac{1}{8}$ in. in diameter.—*Ans.* 14,922 lbs.

19. A revetement wall 40 feet high and 10 feet thick sustains the pressure of earth of mean quality, having the weight of a cubic foot equal to 100 lbs.; it is required to determine whether or not the wall will stand, taking the weight of a cubic foot to be 120 lbs.—*Ans.* The wall will stand.

20. Required the thickness of the wall so that it may be upon the point of overturning.—*Ans.* 8.7 feet.

21. The moment of inertia of a solid cylinder about its axis of symmetry is

$$I = \tfrac{1}{2}\,\pi\, b\, a^4.$$

22. The moment of inertia of a cone about its axis of symmetry is

$$I = \tfrac{1}{10}\,\pi\, b\, a^4.$$

23. The moment of inertia of a cone about an axis passing through its centre of gravity, and perpendicular to its axis, is

$$I = \tfrac{1}{20}\,\pi\, a^2 b\,\{a^2 + \tfrac{1}{4}\, b^2\}.$$

24. The moment of inertia of a cylinder about an axis passing through its centre of gravity, and perpendicular to its axis of symmetry, is

$$I = \tfrac{1}{4}\,\pi\, b\, a^2\,(a^2 + \tfrac{1}{3}\, b^2).$$

efficient of friction .14. How many times will it revolve before it stops?—1.04 turns.

32. There is a cone, the weight of each cubic foot of which is 164 lbs.; the radius of the base is 3 feet; the vertical height = 4 feet. How many units of work must be expended in overturning this cone? Also, how many units of work must be expended in overturning a cylinder of the same height and of the same quantity of material?—13366.9 for cone; 3992.1 for cylinder.

33. Two uniform beams being connected together at the top, and having the lower ends connected by a tie beam: find the pressures in the directions of the beams, and the tension on the tie beam.

34. In the pulley with friction, given the radius of the pulley $= a$; the radius of the axis $= r$; the limiting angle of resistance $= \theta$; two weights P_1 and P_2 acting on the pulley; P_1 being the preponderating weight; then the space described in the time t is

$$S = \tfrac{1}{2} g \cdot \frac{a\,(P_1 - P_2) - r\,(P_1 + P_2) \sin. \theta}{(P_1 + P_2)\,(a - r \sin. \theta)} \cdot t^2.$$

35. For the centre of gravity of the surface generated by the revolution of a semicycloid about its axis,

$$X = \frac{2a}{3} \cdot \frac{\pi - \frac{8}{15}}{\pi - \frac{4}{3}}.$$

36. Find the centre of gravity of a solid formed by the revolution of a semicycloid about its axis.

TABLES

OF VARIOUS KINDS

USEFUL TO ENGINEERS.

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]25	[illegible]
[illegible]91	[illegible]
[illegible]56	[illegible]
.722	[illegible]
.787	[illegible]45
.853	1.[illegible]24
[illegible]	1.102
[illegible]	[illegible]81
	[illegible]
	[illegible]
	[illegible]75
	[illegible].732
	[illegible].890
	2.047
	2.205
	2.362
	2.520
	2.677
	2.835
	2.992
	3.150

THICKNESS IN PARTS OF AN INCH.								Breadth in Inches, &c.
9/16	5/8	11/16	3/4	13/16	7/8	15/16	1	
.0351	.0391	.0430	.0469	.0508	.0547	.0586	.0625	1/16
.0703	.0781	.0859	.0937	.1016	.1094	.1172	.1250	1/8
.1055	.1172	.1289	.1406	.1523	.1641	.1758	.1875	3/16
.1406	.1562	.1719	.1875	.2031	.2187	.2344	.2500	1/4
.1758	.1953	.2148	.2344	.2539	.2734	.2930	.3125	5/16
.2109	.2344	.2578	.2812	.3047	.3281	.3516	.3750	3/8
.2461	.2734	.3008	.3281	.3555	.3828	.4101	.4375	7/16
.2812	.3125	.3437	.3750	.4062	.4375	.4687	.5000	1/2
.3164	.3516	.3867	.4219	.4570	.4922	.5273	.5625	9/16
.3516	.3906	.4297	.4687	.5078	.5469	.5859	.6250	5/8
.3867	.4297	.4726	.5156	.5586	.6016	.6445	.6875	11/16
.4219	.4687	.5156	.5625	.6094	.6562	.7031	.7500	3/4
.4570	.5078	.5586	.6094	.6601	.7109	.7617	.8125	13/16
.4922	.5469	.6016	.6562	.7109	.7656	.8203	.8750	7/8
.5273	.5859	.6444	.7031	.7617	.8203	.8789	.9375	15/16
.5625	.6250	.6875	.7500	.8125	.8750	.9375	1.000	1
.6328	.7031	.7734	.8437	.9141	.9844	1.055	1.125	1 1/8
.7031	.7812	.8594	.9375	1.016	1.094	1.172	1.250	1 1/4
.7734	.8594	.9453	1.031	1.117	1.203	1.289	1.375	1 3/8
.8437	.9375	1.031	1.125	1.219	1.312	1.306	1.500	1 1/2
.9141	1.016	1.117	1.219	1.320	1.422	1.523	1.625	1 5/8
.9844	1.094	1.203	1.312	1.422	1.531	1.641	1.750	1 3/4
1.055	1.172	1.289	1.406	1.523	1.641	1.758	1.875	1 7/8
1.125	1.250	1.375	1.500	1.625	1.750	1.875	2.000	2
1.195	1.328	1.461	1.594	1.726	1.859	1.992	2.125	2 1/8
1.266	1.406	1.547	1.687	1.828	1.969	2.109	2.250	2 1/4
1.336	1.484	1.633	1.781	1.930	2.078	2.226	2.375	2 3/8
1.406	1.562	1.719	1.875	2.031	2.187	2.344	2.500	2 1/2

TABLE *of the* SECTIONAL AREA *of* SQUARE BAR IRON.

Side of Square.	Area in Square Inches.	Side of Square.	Area in Square Inches.	Side of Square.	Area in Square Inches.
1/16	.0039	1 9/16	2.4414	3 1/8	9.7656
1/8	.0156	1 5/8	2.6406	3 1/4	10.5625
3/16	.0351	1 11/16	2.8476	3 3/8	11.3906
1/4	.0625	1 3/4	3.0625	3 1/2	12.2500
5/16	.0976	1 13/16	3.2851	3 5/8	13.1406
3/8	.1406	1 7/8	3.5156	3 3/4	14.0625
7/16	.1914	1 15/16	3.7539	3 7/8	15.0156
1/2	.2500	2	4.0000	4	16.0000
9/16	.3164	2 1/16	4.2539	4 1/8	17.0156
5/8	.3906	2 1/8	4.5156	4 1/4	18.0625
11/16	.4726	2 3/16	4.7851	4 3/8	19.1406
3/4	.5625	2 1/4	5.0625	4 1/2	20.2500
13/16	.6601	2 5/16	5.3476	4 5/8	21.3906
7/8	.7656	2 3/8	5.6406	4 3/4	22.5625
15/16	.8789	2 7/16	5.9414	4 7/8	23.7656
1	1.0000	2 1/2	6.2500	5	25.0000
1 1/16	1.1289	2 9/16	6.5664	5 1/8	26.2656
1 1/8	1.2656	2 5/8	6.8906	5 1/4	27.5625
1 3/16	1.4101	2 11/16	7.2226	5 3/8	28.8906
1 1/4	1.5625	2 3/4	7.5625	5 1/2	30.2500
1 5/16	1.7226	2 13/16	7.9101	5 5/8	31.6406
1 3/8	1.8906	2 7/8	8.2656	5 3/4	33.0625
1 7/16	2.0664	2 15/16	8.6289	5 7/8	34.5156
1 1/2	2.2500	3	9.0000	6	36.0000

[illegible] of the Weight [illegible] Foot of Round Iron. | [illegible] Areas of Round Iron.

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	17.720
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	18.665
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	19.635
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	20.629
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	21.647
[illegible]	[illegible]	[illegible]	76.2107	[illegible]	[illegible]	5 3/8	22.690
[illegible]	[illegible]	[illegible]	[illegible]	2	3.1416	5 1/2	23.758
[illegible]	[illegible]	[illegible]	[illegible]	2 1/8	3.5465	5 5/8	24.850
[illegible]	[illegible]	[illegible]	87.2501	2 1/4	3.9760	5 3/4	25.967
[illegible]	[illegible]	[illegible]	[illegible]	2 3/8	4.4302	5 7/8	27.108
[illegible]	[illegible]	[illegible]	95.0620	2 1/2	4.9087	6	28.274

of the DECIMAL EQUI-
NTS *of the* FRACTIONAL
S *of* 1 CWT. *increasing by*

000	$\frac{1}{4}$=.25	$\frac{2}{4}$=.5	$\frac{3}{4}$=.75
892	.25892	.50892	.75892
1785	.26785	.51785	.76785
2678	.27678	.52672	.77672
3571	.28571	.53571	.78571
4464	.29464	.54464	.79464
5357	.30357	.55357	.80357
6250	.31250	.56250	.81250
7142	.32142	.57142	.82142
8035	.33038	.58035	.83035
8928	.33928	.58928	.83928
9821	.34821	.59821	.84821
0714	.35714	.60714	.85714
1607	.36607	.61607	.86617
2500	.37500	.62500	.87500
3392	.38392	.63392	.88392
4285	.39285	.64285	.89285
5180	.40180	.65180	.90180
6071	.41071	.66070	.91070
6964	.41964	.66964	.91964
7860	.42860	.67860	.92860
8750	.43750	.68750	.93750
9642	.44642	.69642	.94642
0535	.45535	.70535	.95535
1430	.46430.	71430	.96430
2321	.47321	.72321	.97321
3214	.48214	.73214	.98214
4110	.49110	.74110	.99110

TABLE *of the* DECIMAL EQUIVALENTS *of the* FRACTIONAL PARTS *of* 1 INCH, *increasing by* $^1/_{32}$ *inch.*

Inch.		Inch.	
0	.00000	$^1/_2$	.50000
$^1/_{32}$	.03125	$^{17}/_{32}$	.53125
$^1/_{16}$	.06250	$^9/_{16}$	.56250
$^3/_{32}$	.09375	$^{19}/_{32}$	.59375
$^1/_8$	.12500	$^5/_8$	.62500
$^5/_{32}$	.15625	$^{21}/_{32}$	.65625
$^3/_{16}$	.18750	$^{11}/_{16}$	.68750
$^7/_{32}$	.21875	$^{23}/_{32}$	.71875
$^1/_4$	.25000	$^3/_4$	.75000
$^9/_{32}$	.28125	$^{25}/_{32}$	.78125
$^5/_{16}$	.31250	$^{13}/_{16}$	.81250
$^{11}/_{32}$	.34375	$^{27}/_{32}$	.84375
$^3/_8$	.37500	$^7/_8$	.87500
$^{13}/_{32}$	.40625	$^{29}/_{32}$	.90625
$^7/_{16}$	.43750	$^{15}/_{16}$	.93750
$^{15}/_{32}$	.46875	$^{31}/_{32}$	.96875

Diameter in Inches.	Area in Inches.	Diameter in Inches.	Area in Inches.	Diameter in Inches.	Area in Inches.	Diameter in Inches.	Area in Inches.
53	2206.18	60	2827.44	67	3525.66	74	4300.85
53½	2248.01	60½	2874.56	67½	3578.47	74½	4359.16
54	2290.22	61	2922.47	68	3631.68	75	4417.87
54½	2332.83	61½	2970.38	68½	3685.29	75½	4476.97
55	2375.83	62	3019.07	69	3739.29	76	4536.47
55½	2419.22	62½	3067.96	69½	3793.67	76½	4596.25
56	2463.01	63	3117.25	70	3848.46	77	4656.63
56½	2507.19	63½	3166.92	70½	3903.60	77½	4717.31
57	2551.76	64	3217	71	3959.20	78	4778.37
57½	2596.53	64½	3267.46	71½	4015.16	78½	4839.73
58	2642.08	65	3318.31	72	4071.51	79	4901.68
58½	2687.83	65½	3369.56	72½	4128.25	79½	4963.92
59	2733.97	66	3421.20	73	4185.39	80	5026.56
59½	2780.31	66½	3473.23	73½	4242.13		

PARALLEL MOTION TABLES,
When the Length of the Stroke is not taken into consideration.

Radius of Beam in Inches.	Length of Parallel Bar in Inches.	Length of Radius Rod in Inches.	Radius of Beam in Inches.	Length of Parallel Bar in Inches.	Length of Radius Rod in Inches.
72	18	162	78	27	96.5
—	21	123.9	—	30	76.8
—	24	96	—	33	61.36
—	27	75	—	36	49
—	30	58.8	—	39	39
—	33	46	—	42	30.857
—	36	36	—	45	24.2
—	39	27.9	—	48	18.75
—	42	21.428	—	51	14.3
—	45	16.2	—	54	10.666
—	48	12	—	57	7.7
—	51	8.6	—	60	5.4
—	54	6	84	18	242
78	18	200	—	21	189
—	21	154.7	—	24	150
—	24	121.5	—	27	120.3

Radius of Beam in Inches.	Length of Parallel Bar in Inches.	Length of Radius Rod in Inches.	Radius of Beam in Inches.	Length of Parallel Bar in Inches.	Length of Radius Rod in Inches.
114	63	41.3	132	72	50
—	66	34.9	—	75	43.32
—	69	29.3	—	78	37.384
—	72	24.5	—	81	32.111
—	75	20.3	—	84	27.428
120	42	144.8	—	87	23.27
—	45	125	—	90	19.6
—	48	108	—	93	16.35
—	51	93.35	138	48	168.75
—	54	80.666	—	51	148.4
—	57	69.6	—	54	130.666
—	60	60	—	57	115
—	63	51.5	—	60	101.4
—	66	44.181	—	63	89.3
—	69	37.7	—	66	78.545
—	72	32	—	69	69
—	75	27	—	72	60.5
—	78	22.6	—	75	52.9
—	81	18.77	—	78	46.153
126	42	168	—	81	40
—	45	145.8	—	84	34.643
—	48	126.75	—	87	30
—	51	110.3	—	90	25.6
—	54	96	—	93	21.7
—	57	83.5	144	48	192
—	60	72.6	—	51	169.6
—	63	63	—	54	150
—	66	54.545	—	57	132.7
—	69	47	—	60	117.6
—	72	40.5	—	63	104.14
—	75	34.7	—	66	92.181
—	78	29.33	—	69	81.5
—	81	25	—	72	72
—	84	21	—	75	63.5
—	87	17.48	—	78	55.846
132	48	147	—	81	49
—	51	128.6	—	84	42.857
—	54	112.666	—	87	37.35
—	57	98.7	—	90	32.4
—	60	86.4	—	93	28
—	63	75.5	—	96	24
—	66	66	—	99	20.4
—	69	57.5			

Table *of* Plane Surfaces—(*continued.*)

Surfaces in contact.	Disposition of the Fibres.	State of the Surfaces.	Co-efficient of Friction.
Experiments of M. Morin—*continued.*			
Ox-hide as a piston-sheath upon cast-iron	flat or sideways	steeped in water	0.62
		with oil, tallow, or hog's lard	0.12
Black dressed leather, or strap leather, upon a cast-iron pulley . .	flat	without unguent	0.28
		steeped	0.38
Cast-iron upon cast-iron	ditto	without unguent	0.16
Iron upon cast-iron . .	ditto	ditto	0.19
Oak, elm, yoke elm, iron, cast-iron, and brass, sliding two and two, one upon another . .	ditto	with tallow	0.10
		with oil, or hog's lard	0.15
Calcareous oolite stone upon calcareous oolite	ditto	without unguent	0.74
Hard calcareous stone, called muschelkalk, upon calcareous oolite	ditto	ditto	0.75
Brick upon calcareous oolite	ditto	ditto	0.67
Oak upon calcareous oolite	wood endways	ditto	0.63
Iron upon calcareous oolite	flat	without unguent	0.49
Hard calcareous stone, or muschelkalk, upon muschelkalk . . .	ditto	ditto	0.70
Calcareous oolite stone upon muschelkalk .	ditto	ditto	0.75
Brick upon muschelkalk	ditto	ditto	0.67
Iron upon muschelkalk.	ditto	ditto	0.42
Oak upon muschelkalk .	ditto	ditto	0.64
Calcareous oolite stone upon calcareous oolite	ditto	with a coating of mortar, of three parts of fine sand, and one part of slack lime	0.74

TABLE *of* FRICTION *of* PLANE SURFACES—(*continued.*)

Surfaces in contact.	Disposition of the Fibres.	State of the Surfaces.	Co-efficient of Friction.
EXPERIMENTS OF M. MORIN —*continued,*			
Hemp, in threads or in cord, upon oak. . .	parallel	without unguent	0.52
	perpendicular	with water	0.33
Oak and elm upon cast-iron	parallel	without unguent	0.38
Wild pear-tree, ditto .	ditto	ditto	0.44
Iron upon iron . . .	ditto	ditto	0.44
Iron upon cast-iron and brass	ditto	ditto	0.18
Cast-iron, ditto . . .	ditto	ditto	0.15
Brass upon brass . .	ditto	ditto	0.20
Brass upon cast-iron .	ditto	ditto	0.22
Brass upon iron . .	ditto	ditto	0.16
Oak, elm, yoke elm, wild pear, cast-iron, wrought iron, steel, and moving one upon another, or on themselves . . .	ditto	greased in the usual way with tallow, hog's lard, oil, soft gom	0.07 to 0.08
		slightly greasy to the touch	0.15
Calcareous oolite stone upon calcareous oolite	ditto	without unguent	0.64
Calcareous stone, called muschelkalk, upon calcareous oolite . . .	ditto	ditto	0.67
Common brick upon calcareous oolite . .	ditto	ditto	0.65
Oak upon calcareous oolite	wood endways	ditto	0.38
Wrought iron, ditto . .	parallel	ditto	0.69
Calcareous stone, called muschelkalk, upon muschelkalk . . .	ditto	ditto	0.38
Calcareous oolite stone upon muschelkalk .	ditto	ditto	0.65
Common brick, ditto .	ditto	ditto	0.60
Oak upon muschelkalk.	wood endways	ditto	0.38
Iron upon muschelkalk	parallel	ditto	0.24
		saturated with water	0.30

BY THE SAME AUTHOR.

A SHORT TREATISE

ON

THE STEAM ENGINE,

Adapted to the Use of Schools,

IN WHICH ARE GIVEN

PRACTICAL RULES FOR THE USE OF ENGINEERS.

OPINIONS OF THE PRESS.

"'Short,' but good. We do not think that we shall be exaggerating the merits of this Work if we say, that, compared with all previous Works on the subject, whether short or long, it has not its equal."—*Mechanic's Magazine, May* 15.

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